Learning and Payoff Externalities in an Investment Game*

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Abstract

This paper examines the interplay of informational and payoff externalities in a two-player irreversible investment game. Each player learns about the quality of his project by observing a private signal and the action of his opponent. I characterize the unique symmetric equilibrium in a timing game that features a second-mover advantage, allowing for arbitrary correlation in project qualities. Despite private learning, the game reduces to a stochastic war of attrition. In contrast to the case of purely informational externalities, equilibrium behavior displays waves of investments—irrespective of the sign of the correlation—and beliefs never get trapped in a no-learning region, provided that the second-mover advantage is sufficiently high.

Keywords: irreversible investment, payoff externalities, war of attrition, real option.

JEL Codes: D83, D82

1 Introduction

Initial delay in adoption is a commonly observed empirical regularity in the diffusion of a new technology (see Hoppe, 2002). The theoretical literature on innovation dynamics has typically proposed two alternative explanations of this stylized fact: informational spillovers and externalities in the innovation diffusion process. Recently, the strategic experimentation literature has focused on the role of informational spillovers in dynamic environments in which information accumulates over time: the interaction between private and observational learning generates strategic incentives to delay risky ventures and wait for new information.

The objective of this paper is to study the optimal timing of the adoption of a new technology and to analyze the interplay between informational and payoff externalities in the

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form of a second-mover advantage. As in the social learning literature, informational externalities are generated by the possibility of gathering private information prior to investing. However, returns to adoption are determined not only by the uncertain profitability of the new technology but also by the presence of other adopters, giving rise to payoff externalities. A second-mover advantage arises when a follower can free ride on the leader's effort to jump-start the new technology or benefit from better positioning, lower adoption costs, and network externalities. For example, in pharmaceutical markets, later movers can free ride on pioneers' endeavors to increase consumers' perception of the safety and efficacy of a therapeutic class of drugs (see Azoulay et al., 2003). A second-mover advantage also arises when firms make decisions concerning entry into a market with horizontally differentiated products (see Frisell, 2003).

I study the interaction between payoff and informational externalities in a two-player timing game. Each player has the option of undertaking an irreversible investment (e.g., adopting a new technology). Players' investment opportunities are arbitrarily correlated, and each player prefers to be a follower rather than a leader. Over time, each player observes private signals that bring conclusive news about the low quality of his own project, which makes it unprofitable to invest.

First, I show that in the unique symmetric equilibrium, the presence of a second-mover advantage increases investment delay relative to the case with no payoff externalities: players' incentive to wait is now twofold. Waiting increases the probability of benefiting from observational learning and from the second-mover advantage. The structure of the equilibrium is intuitive: the game reduces to a war of attrition with incomplete information in which the payoffs of the leader and follower are specified to capture both payoff and informational externalities.

Then, I investigate how payoff externalities affect the equilibrium outcome and the learning dynamics as compared to the case of purely informational externalities. I derive two predictions in terms of observable variables. First, the probability of investing in an unprofitable project is decreasing in the magnitude of payoff externalities. Second, when projects are negatively correlated, the equilibrium involves waves of investment only if there is a second-mover advantage.

Related Literature

My model combines a timing game of new technology adoption (e.g., Fudenberg and Tirole, 1985 or Reinganum, 1981) under uncertainty and an exponential two-armed bandit problem, as studied by Keller, Rady, and Cripps (2005).

Within the strategic experimentation literature, the paper is closely related to Rosenberg, Salomon, and Vieille (2013) and Murto and Välimäki (2011). These two papers analyze in continuous and in discrete time, respectively, a symmetric two-armed bandit game in which the risky arms are correlated and experimentation outcomes are private, whereas the decision to switch from the risky arm to the safe one is observable and irreversible. Such a game is equivalent to the irreversible investment game I study. However, their analysis is conducted in a setting of purely informational externalities.

The idea of second-mover advantage in an investment game has been explored in Décamps and Mariotti (2004) (see also Kwon, Xu, Agrawal, and Muthulingam, 2016). These authors study a duopoly model of investment in which players learn about the quality of a common value project by observing some public information. In their model, information is asymmetric because each player's investment cost is privately known, while informational externalities arise because the leader's payoffs are observable. As a consequence, private information is gradually revealed as players learn about the investment opportunity, and the increase in signal quality enjoyed by the follower endogenously creates a second-mover advantage.

Recently, other papers have studied the interaction between private learning and irreversible decisions in bandit games that feature payoff and informational externalities. In Akcigit and Liu (2016), two players compete to be the first to achieve a breakthrough on a safe and a risky research line; the paper studies the inefficiencies arising when breakthroughs are observable, while breakdowns (dead ends) and research activities are not. Payoff externalities are positive in the collaboration model of Guo and Roesler (2017) in that players share the proceeds from a breakthrough. Over time, players choose the level at which to exert effort and have the option to abandon experimentation irreversibly and take an outside option. The paper studies effort dynamics when breakthroughs and exit decisions are observable, and (costless) breakdowns and effort are not.

At a broader level, the paper is related to the early literature on herding and observational learning, which assumes that players receive private information about a common state variable at the beginning of the game. As in Chamley and Gale (1994), informational externalities generate strategic delay in investment.

2 The Model

Time is continuous. Each of two players $i \in \{1, 2\}$ chooses when, if ever, to irreversibly invest in a risky project. Each player's project can be either good (G) or bad (B). I denote with $(\omega_1, \omega_2) \in \{G, B\} \times \{G, B\}$ the project-type profile. Players share a symmetric prior distribution over type profiles that attaches probability $p_0 \in (0, 1)$ to each of the events $\{\omega_i = G\}$ for $i = 1, 2$. The prior distribution attaches probability $p_0 \, (p_0 + (1 - p_0)\rho)$ to the event $\{\omega_1 = G, \omega_2 = G\}$ and probability $(1 - p_0)\,(1 - p_0 + p_0\rho)$ to the event $\{\omega_1 = B, \omega_2 = B\}$, with $\rho \in (\max\{-p_0/(1 - p_0), -(1 - p_0)/p_0\}, 0) \cup (0, 1]$.[1] (Equivalently, ρ is the Pearson correlation coefficient for the random variables $\mathbf{1}_{\{\omega_i = G\}}$, $i = 1, 2$.)

Over time, each player receives private signals about the quality of his project. If his project is bad, a player's private signals arrive according to a Poisson process with intensity

[1] That is, the joint distribution over type profiles is

$$\begin{array}{cc} & \begin{array}{cc} G & B \end{array} \\ \begin{array}{c} G \\ B \end{array} & \begin{pmatrix} p_0\,(p_0 + (1 - p_0)\rho) & p_0(1 - p_0)(1 - \rho) \\ p_0(1 - p_0)(1 - \rho) & (1 - p_0)\,(1 - p_0 + p_0\,\rho) \end{pmatrix}. \end{array}$$

$\lambda > 0$. Signal processes are conditionally independent across players and independent of investment decisions. A player never receives any signal if his project is good. Hence, any signal provides conclusive evidence that the project is bad. I denote with $\tau_i \in [0, +\infty]$ the time at which player i observes his first signal.

The game consists of two stages. In the first stage, each player i chooses an investment time $t_i(\varnothing) \in [0, +\infty]$, with the interpretation that player i invests at $t_i(\varnothing)$ if $\tau_i \geq t_i(\varnothing)$, provided that the first stage does not end before $t_i(\varnothing)$. The first stage ends as soon as one player invests, that is, it ends at $\theta := \inf_i\{t_i(\varnothing) + \mathbf{1}_{\{\tau_i < t_i(\varnothing)\}}\infty\}$. Define the set of first movers $I^* := \{i : t_i(\varnothing) = \theta \text{ and } t_i(\varnothing) < \infty\}$.

If exactly one player invests as a first mover, at $\theta < \infty$, the game transitions to the second stage. In the second stage, the second mover $i \notin I^*$ chooses an investment time $t_i(\theta) \in [\theta, \infty]$, with the interpretation that player i invests at $t_i(\theta)$ if $\tau_i \geq t_i(\theta)$.

Players discount their payoffs at a common discount rate $r > 0$. A terminal history specifies the time of the first investment, $\theta \in [0, \infty]$, the set of first movers $I^* \subseteq \{1, 2\}$, and the investment time of the second mover, provided that $\theta < \infty$ and $I^* \neq \{1, 2\}$. Given a terminal history, the payoff to player $i \in I^*$ is

$$e^{-r\theta}\left(L(\omega_i)\mathbf{1}_{\{|I^*|=1\}} + M(\omega_i)\mathbf{1}_{\{|I^*|>1\}}\right).$$

The payoff to player $i \notin I^*$ is $e^{-rt_i}F(\omega_i)$, where $t_i \in [\theta, \infty)$ is the time at which he invests, and is equal to zero if i never invests.

I assume that there is a second-mover advantage, that is,

$$L(\omega_i) \leq M(\omega_i) \leq F(\omega_i), \qquad \omega_i = B, G,$$

with the last inequality being strict for $\omega_i = G$. Further, investing is always profitable if $\omega_i = G$, that is, $L(G) > 0$, and it is unprofitable whenever $\omega_i = B$, that is, $F(B) < 0$. The investment cost is normalized to zero.[2]

A pure strategy is a function

$$t_i : \{\varnothing\} \cup [0, \infty] \to [0, \infty],$$

such that $t_i(t) \geq t$ whenever $t \neq \varnothing$. A (behavior) strategy maps each $t \in \{\varnothing\} \cup [0, \infty]$ to a probability distribution over $[0, \infty]$ with support contained in $[t, \infty]$ whenever $t \neq \varnothing$. I state the results in terms of (symmetric) perfect Bayesian equilibria.

It is worth noting that in this game, any Nash equilibrium is outcome-equivalent to a perfect Bayesian equilibrium. In fact, because every history in which there is no investment is on the equilibrium path, the only observable deviations are those in which a player invests

[2]Assuming that $L(G) - L(B) = F(G) - F(B)$, the game is equivalent to the following strategic bandit game. Each player faces two arms, a safe arm and a risky arm, and decides when to irreversibly switch from the risky to the safe arm. A bad risky arm never yields any payoff. A good risky arm yields no payoff until a random time, after which it yields a constant payoff flow with present value $(\lambda + r)(L(G) - L(B))/\lambda$. The random time follows an exponential distribution with parameter λ. The safe arm yields a constant payoff flow with present value $L(G)$ to the player who first stops experimenting and a payoff flow with present value $F(G)$ to the one who stops second.

when he is supposed to wait. However, by definition, a player's expected payoff from such a deviation does not depend on the opponent's continuation play: a player's terminal payoff is determined at the time of his investment, and once he invests, he becomes inactive. As a result, the equilibrium play remains optimal irrespective of the specification of players' beliefs and continuation strategies after such off-path histories. For convenience, I shall omit the specification of players' beliefs and behavior after off-path histories from the description of the equilibrium.

3 Equilibrium Analysis

3.1 Beliefs

Over time, each player forms beliefs about the profitability of his project. In the first stage, two forces drive the evolution of a player's beliefs: private learning and observational learning. On the one hand, a player's private signal provides conclusive evidence that his project is bad, and the absence thereof makes him optimistic about its quality. On the other hand, a player also learns about the profitability of his own project by observing the action of the opponent: the longer the opponent waits, the more likely it is that he has observed a signal, and hence, his project is bad.

Fix a first-stage strategy $\sigma_j(\varnothing)$ for player j, where $\sigma_j(\varnothing)$ is a probability distribution over $[0, \infty]$. It is convenient to represent such a strategy using the distribution function $G_j : [0, \infty] \to [0, 1]$, defined as $G_j(t) \coloneqq \sigma_j(\varnothing)([0, t])$. Let $p_i(t)$ be player i's belief about his own project if he has not observed any signal, and player j has not invested by time t, i.e.,

$$p_i(t) \coloneqq \Pr[\omega_i = G \mid \theta \geq t, \tau_i \geq t] \,.$$

The map $p_i : [0, \infty) \to [0, 1]$ is left-continuous and admits a right limit. The discontinuities of $p_i(t)$ coincide with the atoms of the distribution $\sigma_j(\varnothing)$. By Bayes' rule, whenever differentiable, $p_i(t) \in (0, 1)$ solves

$$\begin{aligned} \frac{p_i'(t)}{p_i(t)\,(1 - p_i(t))} &= \lambda \\ &\quad - \left(\Pr\left[\tau_j > t \mid \omega_i = G, \theta \geq t\right] - \Pr\left[\tau_j > t \mid \omega_i = B, \theta \geq t\right]\right) \frac{G_j'(t)}{1 - G_j(t)} \,. \end{aligned} \tag{1}$$

The differential equation elucidates the two forms of learning: private and observational learning. First, as time passes, player i grows optimistic since he has not observed any signal: this is captured by the term λ on the right-hand side of (1). Second, player j not having invested yet may be evidence of his project being bad, which brings either good or bad news, depending on the sign of ρ. It can be shown that the term in parentheses in (1) has the same sign as ρ: plainly, when the projects are positively (negatively) correlated, for any given history, the opponent is more likely to be uninformed if a player's own project is good (bad).

To put it differently, over time, each player forms beliefs about the quality of his own project, about whether the opponent has observed a signal, and about the quality of opponent's project. In the special case of perfect positive correlation, $\rho = 1$, equation (1) reduces to

$$\frac{p_i'(t)}{p_i(t)\,(1-p_i(t))} = \lambda - \left(\frac{1-\Pr\left[\tau_j > t \mid \theta \geq t, \tau_i \geq t\right]}{1-p_i(t)}\right)\frac{G_j'(t)}{1-G_j(t)}\,.$$

When the correlation is perfect, at any point in time from the point of view of player i, the payoff-relevant uncertainty is summarized by his belief about the common quality of the projects and his belief about the information held by player j.

3.2 Single-Agent Problem

I start with the problem of a single player who observes only his own signal, which is a natural benchmark and corresponds to the problem faced by a second mover. In this section, I describe the optimal investment policy in the single-agent problem in which the net present value payoff from investing as a function of the quality of the project is $F(\omega)$. Throughout the section, I shall omit the player's subscript.

Fix an initial belief, and denote by $\pi(t) \in [0,1)$ the belief held by the player at time t if he has not received any signal, that is, $\pi(t) \coloneqq \Pr[\omega = G \mid \tau \geq t]$, where τ denotes the time of his first signal. By Bayes' rule, the posterior belief evolves according to

$$\pi'(t) = -\lambda\pi(t)\,(1-\pi(t))\ .$$

By the principle of optimality, the value function satisfies, to the first order:

$$w(\pi) = \max\left\{\pi F(G) + (1-\pi)F(B), e^{-r\,\mathrm{d}t}\mathbb{E}\left[w(\pi + \mathrm{d}\pi) \mid \pi\right]\right\}\ , \tag{2}$$

where the first term of the maximum is the expected payoff from investing and the second term is expected payoff from waiting.

As in standard real option problems, the agent faces a tradeoff between waiting for information and delaying investment. Because information accumulates over time, as an agent waits longer, the probability of investing in a bad project decreases. However, waiting is costly due to discounting.

When waiting is optimal, so that the maximum in (2) is achieved by the second term, the value function satisfies the differential equation

$$w(\pi)(r + (1-\pi)\lambda) = w'(\pi)((1-\pi)\pi\lambda)\,,$$

with solution $w(\pi) = (\pi/(1-\pi))^{r/\lambda}\,\pi\,K$, where $K \in \mathbf{R}$ is a constant to be determined. Following Keller et al. (2005) (see also Décamps and Mariotti, 2004), I apply value-matching and smooth-pasting to obtain the following lemma, whose proof is omitted. Optimality follows by standard verification arguments, as in Proposition 3.1 of Keller et al. (2005).

Lemma 1. *In the single-agent problem, there is a cutoff belief* p^* *given by*

$$p^* = \frac{-(r+\lambda)F(B)}{rF(G) - (r+\lambda)F(B)},$$

such that below the cutoff, it is optimal for the player to wait, and above it, it is optimal to invest. As a function of the belief π*, the value function is given by*

$$W(\pi) = \left(\frac{\pi}{1-\pi}\frac{1-p^*}{p^*}\right)^{r/\lambda} \frac{\pi}{p^*}\left(p^*F(G) + (1-p^*)F(B)\right),$$

when $\pi < p^*$*, and* $W(\pi) = \pi F(G) + (1-\pi)F(B)$ *otherwise.*

Let p_* be the optimal cutoff belief in the single-agent problem in which the net present value payoff from investing is $L(\omega)$,

$$p_* := \frac{-(r+\lambda)L(B)}{rL(G) - (r+\lambda)L(B)}.$$

and define $t_* := \min\{t \geq 0 : \pi(t) \geq p_*\}$.

3.3 Symmetric Equilibrium

In any equilibrium, the second mover behaves according to the optimal single-player policy described in Lemma 1. At θ, he updates his belief about the profitability of his project, taking correlation into account. The probability that the second mover i attaches to $\{\omega_i = G\}$ at θ after he has seen the other player investing is either 0 or $\phi(\theta) := \Pr[\omega_i = G \mid \tau_i \geq \theta, \tau_j \geq \theta]$, depending on whether $\tau_i < \theta$ or $\tau_i \geq \theta$.[3] In the first case, the second mover finds it optimal never to invest. In the second case, the expected continuation payoff equals $W(\phi(\theta))$, where $W : [0,1] \to \mathbb{R}$ is defined in Lemma 1; the second mover invests as soon as his posterior belief about the quality of his project coincides with the single-player cutoff p^*, or immediately, if $\phi(\theta) \geq p^*$.

In the following, I describe equilibria by specifying only the profile of strategies adopted by players in the first stage. That is, I say that (σ_1, σ_2) is an equilibrium if the profile of strategies according to which $\sigma_i(\varnothing) = \sigma_i$ and players behave in the second stage as explained in the previous paragraph is an equilibrium.

The next result characterizes the unique symmetric equilibrium of the game. With a slight abuse of notation, define the first- and second-mover payoff functions as, respectively, $L(p) := pL(G) + (1-p)L(B)$ and $F(p) := pF(G) + (1-p)F(B)$.

[3]More precisely, beliefs are updated according to Bayes' rule on path, i.e., whenever θ is in the support of the equilibrium strategy. As discussed, the specification of off-the-equilibrium path beliefs and behavior is inconsequential as far as the equilibrium is concerned and is hence omitted.

Theorem 1. *There is a unique symmetric equilibrium, which is nonatomic. The cdf of the equilibrium strategy* $G : [0, \infty) \to [0, 1]$ *is*

$$G(t) = \begin{cases} 0 & t \leq t_*, \\ \hat{G}(t) & t \in (t_*, \bar{t}), \\ 1 & t \geq \bar{t}, \end{cases}$$

where $\hat{G}(t)$ *is the unique solution of the following integro-differential equation*[4] *on* $[t_*, \infty)$ *such that* $\hat{G}(t_*) = 0$

$$\begin{aligned} (\phi(t) - p_*)\,(1 - G(t)) + (1 - \phi(t))\, e^{\lambda t} \left(1 - e^{-\lambda t_*} + \lambda \int_{t_*}^{t} e^{-\lambda s}(1 - G(s))\, \mathrm{d}s\right) \\ \cdot \left(\Pr[\omega_i = G \mid \omega_j = B, \tau_i \geq t] - p_*\right) = p_* \left(\frac{W(\phi(t)) - L(\phi(t))}{-(\lambda + r)L(B)}\right) G'(t), \end{aligned} \tag{3}$$

and $\bar{t} := \inf\{t : \hat{G}(t) = 1\}$.

Equation (3) characterizes each player's investment behavior conditional on him not having observed any signal and the game being in the first stage. By the equilibrium condition, at any time t in the support of the equilibrium distribution, $t \in [t_*, \bar{t})$, player i is indifferent between investing at t and waiting an additional arbitrarily short amount of time before investing. Let θ_j be the (random) time at which player j invests. The indifference condition of player i at time t can be written as

$$\lim_{\varepsilon \to 0} \frac{\Pr[\theta_j < t + \varepsilon \mid \theta_j \geq t, \tau_i \geq t]}{\varepsilon} = \frac{rL(p_i(t)) + \lambda(1 - p_i(t))L(B)}{W(\phi(t)) - L(\phi(t))}. \tag{4}$$

(Recall that by assumption, $W(\phi(t)) - L(\phi(t)) > 0$ for any $\phi(t) \in (0, 1]$.)

The left-hand side is the probability that player i attaches to player j investing in the interval $[t, t + \varepsilon)$. Assuming that player j invests according to the (absolutely continuous) distribution function G, the "expected" investment rate in (4) is related to the distribution function by

$$\lim_{\varepsilon \to 0} \frac{\Pr[\theta_j < t + \varepsilon \mid \theta_j \geq t, \tau_i \geq t]}{\varepsilon} = \Pr[\tau_j \geq t \mid \theta_j \geq t, \tau_i \geq t]\, \frac{G'(t)}{1 - G(t)}.$$

Equation (4) makes it clear that the game reduces to a stochastic war of attrition: the expected investment rate that makes player i indifferent between waiting and investing (the left-hand side) equals the ratio between the rate at which the expected fist-mover payoff decreases and the second-mover advantage (the right-hand side).

[4]Note that $\Pr[\omega_i = G \mid \omega_j = B, \tau_i \geq t] = (p_0(1 - \rho)) \,/\, \left(p_0(1 - \rho) + (1 - p_0(1 - \rho))\, e^{-\lambda t}\right)$ and hence is a function only of time.

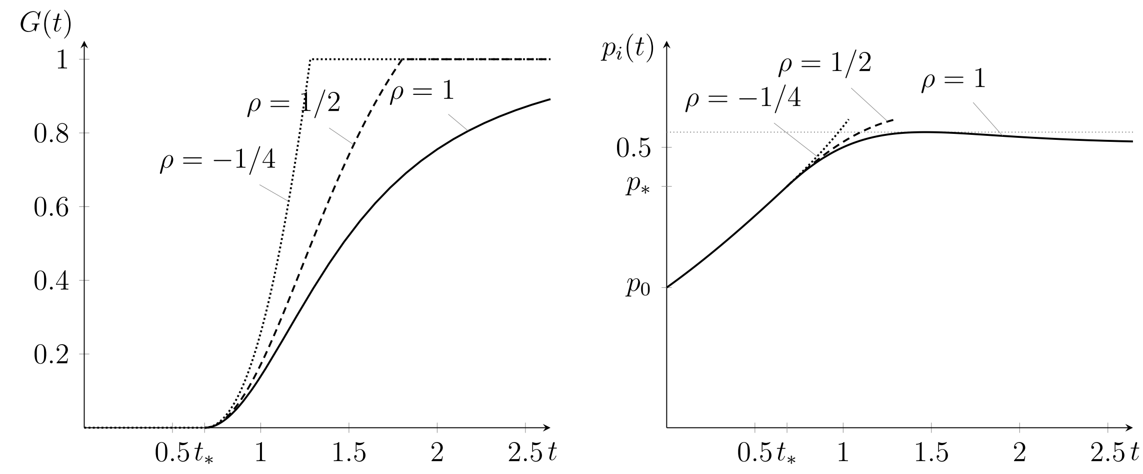

Figure 1: Equilibrium distribution $G(t)$ (left) and belief trajectories $p_i(t)$ (right) for $(p_0, \lambda, r, L(G), L(B), F(G), F(B)) = (1/4, 6/5, 1/2, 9, -2, 10, -1)$

However, in contrast to a stochastic war of attrition in which information is public, players' actions reveal their private information. As a result, while the belief $p_i(t)$ appears in the expected first-mover payoff (the numerator in (4)), the expected second-mover advantage (the denominator in (4)) is a function of $\phi(t)$, the belief conditional on none of the players having observed any signal.

The upshot of Theorem 1 is that the existence and uniqueness of a symmetric equilibrium proven by Rosenberg et al. (2013) is robust to the introduction of payoff externalities. Although the equilibrium characterization in terms of a stochastic war of attrition is specific to my model because it relies on positive payoff externalities, there is no discontinuity either in the equilibrium (first-stage) strategy or in the belief path.

Figure 1 illustrates the equilibrium distribution (left panel) and the belief trajectory (right panel) for different levels of correlation. In the case of positive correlation, the belief trajectory need not be monotone. In fact, the equilibrium is not Markovian in this variable because a player's belief about his own project does not suffice to summarize his assessment of the different sources of uncertainty.

Figure 1 also provides an illustration of a general equilibrium feature. The support of the equilibrium strategy is bounded or unbounded depending on whether the projects' qualities are perfectly correlated.

Lemma 2. $\bar{t} = \infty$ *if and only if* $\rho = 1$.

The intuition behind Lemma 2 is as follows. At any time t in the support of the equilibrium strategy, player j's delay in investing is interpreted by player i as evidence in support of the event $\{\tau_j < t\}$.

When projects are perfectly correlated, the probability that player i attaches to $\{\tau_j < t\}$ is bounded above by $1 - p_*$. In fact, as shown in the Appendix, irrespective of the correlation, whenever a player finds it optimal to invest, the probability he attaches to his project being bad is no larger than $1 - p_*$. Player j observes a signal only if his project is bad, and with perfect correlation, player j's project is bad only if player i's project is bad, hence the bound.

With imperfect correlation, the longer the delay of player j is, the higher the probability that player i (if uninformed) attaches to the event $\{\omega_i = G, \omega_j = B\}$. In contrast to the case of perfect correlation, if $\rho < 1$, the probability that player i attaches to $\{\tau_j < t\}$ converges to 1 as t approaches $\bar{t}$. A player who is sufficiently confident that the opponent will never invest has no reason to abstain from investing unless he has observed a signal himself, and hence, the support of the equilibrium strategy is bounded. Moreover, it follows from the proof of Lemma 2 that $\lim_{t \to \bar{t}} G'(t) \neq 0$, which explains the kinks in the left panel of Figure 1.

Similar to Rosenberg et al. (2013), the symmetric equilibrium is the unique perfect Bayesian equilibrium if the correlation is negative,[5] while there exist asymmetric equilibria in which the equilibrium strategy involves atoms if the projects are positively correlated. In particular, when $\rho > 0$, there are two pure strategy equilibria, $(t_*, \tilde{t})$ and $(\tilde{t}, t_*)$, where $\tilde{t} := \min\{t \geq 0 : \Pr[\omega_i = G \mid \omega_j = B, \tau_i \geq t] \geq p_*\}$.[6]

4 The Role of Payoff Externalities

The magnitude of the second-mover advantage does not affect welfare in the symmetric equilibrium. Each player's ex ante expected payoff is equal to the value of the single-agent problem with payoff function L. While payoff externalities do not affect welfare, they affect the equilibrium outcome, namely, the distribution of investment times conditional on type profiles, and the learning dynamics.

An increase in the second-mover advantage raises the continuation payoff after the first investment. As equilibrium payoffs do not change, a higher expected payoff in the second stage must be offset by an increase in the investment delay. The following proposition formalizes this intuition. For the purpose of this section, it is convenient to assume $F(G) = L(G) + \Delta$ and $F(B) = L(B) + \Delta$; thus, $\Delta > 0$ measures the magnitude of the second-mover advantage.

Proposition 1. *Fix $L(B)$ and $L(G)$, and let $F(\omega) = L(\omega) + \Delta$. The equilibrium distribution functions are ranked by first-order stochastic dominance: $G(t)$ decreases in Δ, for all $t \geq 0$.*

The result is intuitive but not obvious. In light of the comparative statics in war of attrition games, it is natural to expect that the delay should increase with Δ. However, a player who expects the opponent to wait longer before investing attaches lower probability to the opponent being informed should he not invest. As a result, when the projects are positively correlated, a decrease in the opponent's distribution (in the sense of first-order stochastic dominance) makes a player more optimistic about the quality of his own project, thereby increasing the cost of delaying investment.

[5]Equilibrium uniqueness in the case of negative correlation follows from Rosenberg et al. (2013); since only minor modifications of their arguments are needed, I omit a formal proof.

[6]As in Rosenberg et al. (2013), more complex asymmetric equilibria may exist.

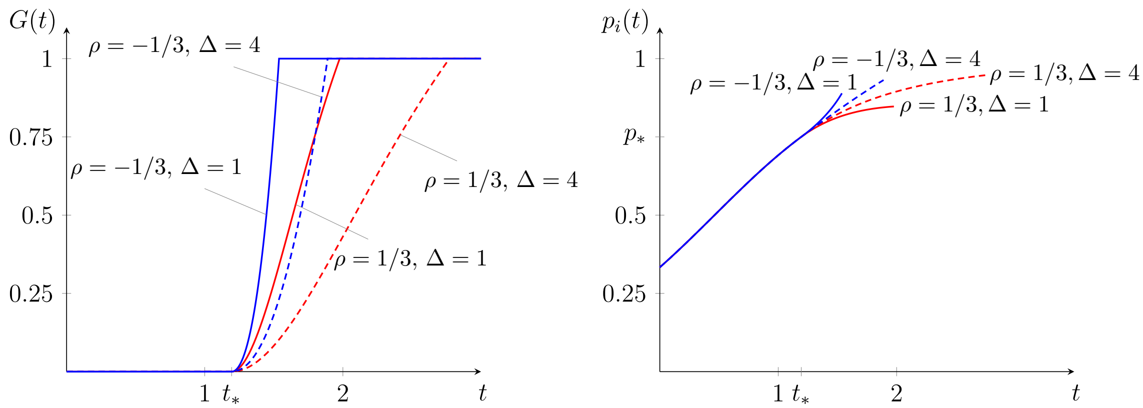

Figure 2: Equilibrium distribution $G(t)$ (left) and belief trajectories (right) for $(p_0, \lambda, r, L(G), L(B), F(G), F(B)) = (1/3, 3/2, 1/2, 8, -6, 8 + \Delta, -6 + \Delta)$

In the proof of Proposition 1, I first show that the equilibrium belief paths for different levels of Δ are ordered pointwise. A larger Δ increases or decreases the belief trajectory depending on whether projects are positively or negatively correlated. Next, I show that the ranking of beliefs implies the ranking of investment distributions. Intuitively, if projects are positively correlated, at a given time, a player can hold a higher belief only if the probability he attached to an uninformed opponent investing before that time is lower. The opposite is true if projects are negatively correlated.

Figure 2 provides an illustration. Proposition 1 has implications in terms of equilibrium outcome and learning dynamics. I explore these implications in the following sections.

4.1 Equilibrium Outcome

Private information accumulates over time. The longer a player delays his investment, the lower the chances of investing in a bad project acting as a first-mover and the stronger the informational content generated by the investment of the opponent, should the latter be the first mover. As a result, the magnitude of the second-mover advantage affects the overall probability of investing in a bad project.

Proposition 2. *In equilibrium, the total probability that player i invests in a bad project,* $\Pr[\theta_i < \infty \mid \omega_i = B]$*, is decreasing in* Δ*.*

While the proposition is valid irrespective of the sign of ρ, a player who acts as a second mover behaves very differently depending on the sign of the correlation. When the correlation is positive, an investment by one player brings good news to the other player, and the equilibrium always exhibits waves of investments. That is, in equilibrium, if only one player invests as a first mover, the opponent follows suit with no delay, i.e., $t_i(t) = t$ for all $t \geq t_*$.

When the correlation is negative, depending on the parameters, the equilibrium may or may not exhibit waves of investments: whether a second mover finds it optimal to wait for a while before investing depends on $\phi(\theta) \lesseqgtr p^*$. Because $\phi(t)$ is increasing in t, a second mover acts with no delay only if the first investment occurs sufficiently late in the game. The next proposition states that when the correlation is negative, the equilibrium exhibits waves of investment only if the second-mover advantage is sufficiently high.

Proposition 3. *For any feasible vector of parameters $(p_0, \rho, \lambda, L(G), L(B))$, there exists $\kappa \in (0, -L(B))$ such that whenever $F(\omega) > L(\omega) + \kappa$ for $\omega = G, B$, in equilibrium the second mover invests at time θ with positive probability.*

The result stands in contrast to the case of purely informational externalities: as shown by Rosenberg et al. (2013), if the correlation is negative, the equilibrium never exhibits waves of investments because $\phi(\bar{t}) < p^*$. In the absence of payoff externalities, a player who is willing to invest as a first mover at time t would postpone action should the opponent invest at time t. In other words, the investment of a player always brings useful information to the other player, who then revises his course of action.

With payoff externalities, if the first investment occurs sufficiently late in the game, an uninformed second mover follows suit: intuitively, the information revealed by the first investment cannot overturn the evidence accumulated up to that time. Yet, even when there are no informational benefits to moving second, because of the second-mover advantage, players are willing to delay investment, that is, $\phi(t) > p^*$ for some $t \in [t_*, \bar{t}]$.

4.2 Learning Dynamics

An implication of Proposition 1 is that the magnitude of the second-mover advantage affects the speed of observational learning. The larger the second-mover advantage is, the weaker the inference drawn by the lack of investment. Whether slower observational learning makes players more or less optimistic at any point in time depends on the sign of the correlation.

Given a strategy of player j, player i's belief at time t satisfies

$$\frac{p_i(t)}{1-p_i(t)} \cdot \frac{1-p_0}{p_0} = \frac{\pi(t)}{1-\pi(t)} \cdot \frac{\Pr[\omega_i = G \mid \theta_j \geq t]}{1-\Pr[\omega_i = G \mid \theta_j \geq t]}, \tag{5}$$

where, as before, $\pi(t) := \Pr[\omega_i = G \mid \tau_i \geq t]$ describes player i's belief based on private signals only.

The first likelihood ratio on the right-hand side of (5) is increasing over time because signals bring bad news, and thus, the absence of signals makes the player more optimistic. When $\rho < 0$, the second likelihood ratio is also increasing. A longer delay by the opponent is good news because it is more likely to occur if the opponent's project is bad. To the contrary, when $\rho > 0$, the second likelihood ratio decreases with time; in fact, the belief trajectory may not be monotone (see Figure 1).

When $\rho > 0$, observational learning and private learning provide confounding evidence. Along a player's history with no signal, the bad news from the delay of the opponent dampens the good news from the lack of private signals. Rosenberg et al. (2013) and Murto and Välimäki (2011) observe that this phenomenon has extreme consequences in the absence of payoff externalities: for any level of positive correlation, the player's equilibrium belief in the first stage of the game is constant. In a sense, the passage of time is uninformative about the state of the world.

As explained by these authors, in the absence of payoff externalities, a player considering delaying his investment by a small $\varepsilon > 0$ faces the same tradeoff as the single agent. In the game, the player may receive an additional piece of news, as the opponent may invest before $t + \varepsilon$. However, if $\rho > 0$, this additional information is confirmatory. That is, it can only trigger investment even before $t + \varepsilon$: there is essentially no additional informational benefit from waiting. As a result, each player's posterior at any time in the support of the equilibrium distribution must be equal to the cutoff belief that makes the single agent indifferent between waiting and investing.

The situation is different when there are payoff externalities. In that case, there is an additional benefit from waiting to be the second mover. As a result, the belief is never constant. Yet, for some parameters, confounded learning may still arise asymptotically.

In the special case of perfectly positively correlated types, the relationship between payoff externalities and confounded learning can be made precise. When $\rho = 1$, the support of the equilibrium distribution is unbounded; in other words, each player attaches positive probability to reaching any first-stage history of arbitrary length. Consequently, the asymptotic belief $\lim_{t\to\infty} p_i(t)$ provides a natural gauge.

The next result establishes when confounded learning arises asymptotically. Define $v := rL(G)/\left(F(G) - L(G)\right)$; namely, v is the investment rate in the unique symmetric equilibrium of the war of attrition in which both projects are known to be good (see Hendricks et al., 1988).

Proposition 4. *Assume $\rho = 1$. In the unique symmetric equilibrium,*

$$\lim_{t\to\infty} p_i(t) = \begin{cases} 1 & \text{if } v < \lambda, \\ \dfrac{-L(B)(\lambda + r)}{(r+\lambda)(L(G) - L(B)) - \lambda F(G)} & \text{if } v \geq \lambda. \end{cases}$$

The belief converges to 1 if and only if the "attrition motive", as captured by v, is sufficiently strong. When the second-mover advantage is small, the interaction of observational and private learning slows the inference process: eventually, the history becomes uninformative about the state, and the posterior belief converges to some interior level.

The condition $v \gtreqless \lambda$ is easy to understand. If beliefs are to converge to 1, the good news from the absence of private signals must eventually dominate the bad news from no investment, so that the equilibrium belief increases over time. As players become arbitrarily optimistic, the indifference condition prescribes investment at a rate arbitrarily close to (but strictly lower than) the complete information concession rate v. If $v \geq \lambda$, namely, the expected investment rate is larger than the rate at which signals arrive, no news is bad news; thus, the belief decreases. Hence, beliefs converge to one only if $v \leq \lambda$.

On the other hand, if players' beliefs are to converge to some interior value, the two informative events must arrive at the same rate. This implies that heuristically, the investment rate must converge to λ so that, in the limit, the belief is constant.

To be clear, the belief $p_i(t)$ is the player's posterior along a history that occurs with vanishing probability as $t \to \infty$. Even when the asymptotic belief converges to an interior value, an outside observer who does not have access to players' private signals becomes increasingly pessimistic about the state of the world, as is evident from (5): because the first likelihood ratio diverges as $t \to \infty$, the second likelihood ratio must converge to zero.

Nevertheless, Proposition 4 sheds light on the main forces at play in the symmetric equilibrium. In short, the history can be informative of the underlying state only if waiting to be the second mover brings useful information and/or payoff benefits. If both of these advantages are absent, an impatient player can be made indifferent between waiting and investing only if time (i.e., a lack of signals and investments) does not bring information either.

5 Extensions

In this section, I discuss the key assumptions of the model and the extent to which they can be relaxed.

5.1 Richer Payoff Interdependence

Assuming that the leader's payoff from investing does not depend on the behavior of the follower avoids the need of discussing refinements. In this section, I argue that this convenient assumption is not critical to the main findings.

In the spirit of Gale (1995), assume that if player $i = 1, 2$ is the first to invest at θ, i.e., $I^* = \{i\}$, and player j invests at $t_j > \theta$, then the realized payoff of player i is

$$r \left(\int_{\theta}^{t_j} e^{-rt} L(\omega_i) \, \mathrm{d}t + \int_{t_j}^{\infty} e^{-rt} F(\omega_i) \, \mathrm{d}t \right),$$

That is, the flow payoff from investing is increasing in the number of investors: the first mover bears a loss in payoff of $\Delta^{\omega} := F(\omega) - L(\omega)$ until the follower joins. The setup is otherwise unchanged.

I focus on the case of positive correlation, $\rho > 0$ and $p_0/(1 - p_0) < -F(B)/F(G)$. Last, I assume that

$$(r + \lambda) \left(L(B) + p_0(1 - \rho) \left(\Delta^G + \Delta^B \right) \right) + 2(r + 2\lambda)(1 - p_0 + p_0\rho)\Delta^B \leq 0,$$

(The role of these assumptions is discussed in the Appendix.)

I focus on (symmetric) perfect Bayesian equilibria satisfying a refinement in the same spirit of those proposed by Cho and Kreps (1987) and Banks and Sobel (1987). Recall that not investing is a dominant action for a player who has observed a signal and that the only observable deviations are those in which a player invests when he is supposed to wait. I require that the off-the-equilibrium-path beliefs after such a deviation attach probability zero to the deviating player having observed a signal prior investing.[7]

Theorem 2. *There is a unique symmetric equilibrium, which is nonatomic. The cdf of the equilibrium strategy* $G : [0, \infty) \to [0, 1]$ *is*

$$G(t) = \begin{cases} 0 & t \leq t^\dagger, \\ \breve{G}(t) & t \in (t^\dagger, \bar{t}), \\ 1 & t \geq \bar{t}, \end{cases}$$

where $t^\dagger < t_*$, $\breve{G}(t)$ *is the unique solution of the integro-differential equation* (19) *such that* $\breve{G}(t^\dagger) = 0$, *and* $\bar{t} := \inf\{t : \breve{G}(t) = 1\}$. *(The definition of* $t^\dagger$ *and the integro-differential equation* (19) *are relegated to the Appendix.)*

Not surprisingly, in equilibrium, players find it optimal to invest earlier, as compared to the baseline model, i.e., $t^\dagger < t_*$. In fact, the expected payoff from investing as a leader is now higher because in equilibrium, if the follower is uninformed, he follows suit, and the leader does not bear any miscoordination costs.

Proposition 1 generalizes to this setup as well: the larger the miscoordination cost is, the longer the delay. (See Figure 3.)

Proposition 5. *Fix* $F(B)$ *and* $F(G)$, *and let* $L(\omega) = F(\omega) - \Delta$. *The equilibrium distribution functions are ranked by first-order stochastic dominance:* $G(t)$ *decreases in* Δ, *for all* $t \geq 0$.

5.2 Second-mover Advantage

While there is an extensive literature on the advantage of being the first to adopt a new technology, a number of empirical studies suggest that second movers can sometimes have an advantage.[8]

Nevertheless, a complete analysis of the case of a first-mover advantage is of interest but beyond the scope of this paper. Here, I describe the challenges that arise in extending the analysis to that case and summarize the similarities and differences with the case of a second-mover advantage.

In the case of a first-mover advantage too, the irreversibility of actions allows for the incomplete information game to be reduced to a stochastic timing game of complete information by appropriately defining a player's payoffs in cases where he is the first mover, where he is the second mover, and where there is simultaneous investment.

[7] To see that other equilibria exist, assume that $\rho = 1$. Note that if a player expects the opponent not to follow suit, then he will not find it optimal to invest earlier than t_*. One can construct a symmetric equilibrium in which no investment occurs earlier than t_* by assuming that following a deviation to early investment, the remaining player attaches probability 1 to the deviating player having observed a bad signal.

[8] Tellis and Golder (1996) show that, on average, second entrants have greater long-term success and higher market shares; in their analysis of two pharmaceutical markets, Shankar et al. (1998) find that second movers enjoy higher profits, and Boulding and Christen (2003) find that pioneers incur larger and persistent cost disadvantages.

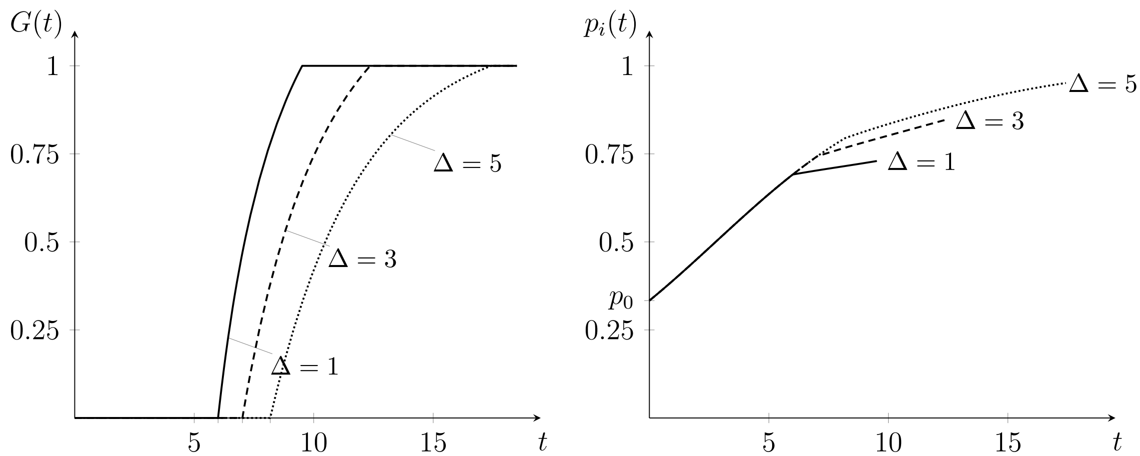

Figure 3: Equilibrium distribution $G(t)$ (left) and belief trajectories $p_i(t)$ (right) for $(p_0, \rho, \lambda, r, L(G), L(B), F(G), F(B)) = (1/3, 2/5, 1/4, 1/2, 6-\Delta, -8-\Delta, 6, -8)$.

One can show that when $\rho \in (0,1)$, there exists a symmetric equilibrium in nonatomic strategies. The equilibrium distribution solves the differential equation (3), but in contrast to the case of a second-mover advantage, investment may occur earlier than in the case of a single agent, and players' beliefs in the first stage are bounded above by p_*. The equilibrium distribution and the belief trajectory in such an equilibrium are shown in Figure 4.

If $\rho \notin (0,1)$ and $M(G) < L(G)$, such an equilibrium does not exist. I suspect that in this case, the existence of a symmetric equilibrium can be guaranteed only by extending the definition of strategies as in Fudenberg and Tirole (1985) (see also Riedel and Steg, 2017) to endow players with an endogenous coordination device.[9] With this extension, there exists an equilibrium in which the first investment occurs at the "boundary" of the preemption region, i.e., as soon as a player is indifferent between being the first or the second mover, and each player becomes a first mover with probability 1/2.[10]

[9]Adopting Riedel and Steg (2017)'s approach in this setup raises some concerns regarding the interpretation. As in Fudenberg and Tirole (1985), the extension aims to represent the limit of a sequence of strategies for discrete-time games with vanishing length of period. Applying the strategy extension to the incomplete information timing game with payoffs $e^{-r\theta} L(\omega_i) \mathbf{1}_{\{i \in I^*, |I^*|=1\}} + e^{-r\theta} M(\omega_i) \mathbf{1}_{\{i \in I^*, |I^*|=2\}} + e^{-r\theta} W(\phi(\theta)) \mathbf{1}_{\{i \notin I^*\}}$ neglects the fact that in the discrete-time game of incomplete information, players update their belief between periods. To circumvent these issues, one could take the alternative route of imposing a simple tie-breaking rule that specifies the probability with which each player becomes a second mover in the case of simultaneous investment.

[10]Incidentally, this equilibrium parallels the unique symmetric equilibrium of the bandit game analyzed by Thomas (2018) in the special case in which switching from the risky arm to the safe one is irreversible (see Thomas, 2018, Sec. 4). In fact, Thomas's game is equivalent to a winner-takes-all irreversible investment game.

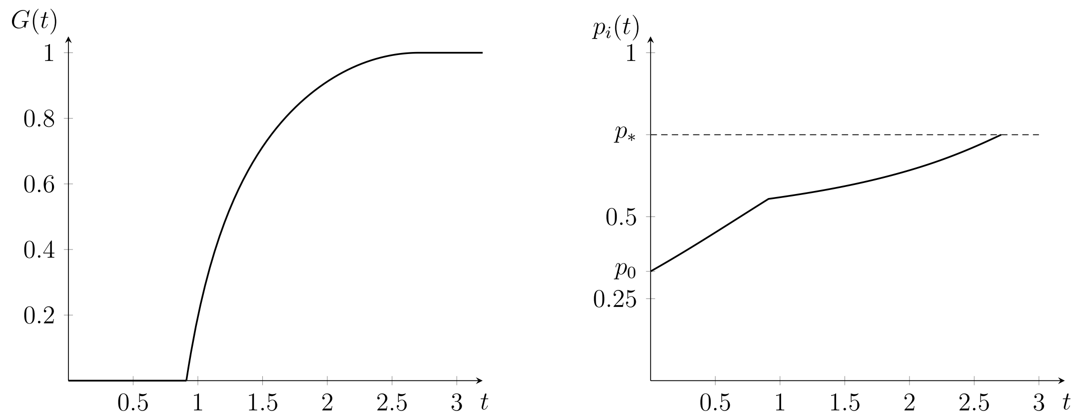

Figure 4: Equilibrium distribution $G(t)$ (left) and belief trajectories $p_i(t)$ (right) in the nonatomic equilibrium for $(p_0, \rho, \lambda, r, L(G), L(B), F(G), F(B)) = (1/3, 1/2, 1, 1/2, 2, -2, 1, -3)$

5.3 Tie-breaking Rule

The uniqueness of the symmetric equilibrium relies on the tie-breaking assumption. If a simultaneous investment yields the second-mover payoff to both players (i.e., $M(\omega) = F(\omega)$ for $\omega = G, B$), the game admits additional symmetric equilibria in which the equilibrium strategy has atoms. The set of symmetric equilibrium payoffs can be characterized by focusing on the simple class of equilibria in which the strategy has at most one atom. Further, the (unique) nonatomic equilibrium achieves the worst equilibrium payoff, while the best equilibrium strikes a balance between delay (the time it takes for players to start investing with positive probability) and coordination (the probability of a simultaneous investment).

5.4 Learning from the Leader's Experience

I assumed that no additional information is generated by the first investment: the only information obtained by the second mover comes from observational learning. However, the payoff structure makes it possible to capture situations in which an additional signal is generated as soon as the first player invests. In the spirit of Décamps and Mariotti (2004) (see also Hoppe, 2000), assume that at the time of the first investment, the second mover observes a public good-news signal. Then, even in the absence of payoff externalities, the expected second-mover payoff function would have the same properties as in the baseline case. Thus, the framework of the paper can be used to study the interaction between observational learning and learning externalities due to information generated by the leader's investment.

Appendix

A.1 Preliminaries

I denote with $u_i(t, \sigma_j)$ the expected payoff induced by the strategy profile (t, σ_j). Here, I derive the explicit formula for $u_i(t, \sigma_j)$.

Let θ_j denote the investment time of player j. Player i's expected payoff is

$$u_i(t, \sigma_j) = \Pr\left[\theta_j > t, \tau_i \geq t\right] e^{-rt} L(p_i(t+)) + \Pr\left[\theta_j = t, \tau_i \geq t\right] e^{-rt} M(\phi(t)) \\ + \mathbb{E}\left[e^{-r\theta_j} W\left(\phi(\theta_j)\right) \mathbf{1}_{\{\theta_j < \min\{t, \tau_i\}\}}\right],$$

where $p_i(t+) \coloneqq \Pr\left[\omega_i = G \mid \theta_j > t, \tau_i \geq t\right]$. For $t_0 < t$,

$$\begin{aligned} & u_i(t, \sigma_j) - u_i(t_0, \sigma_j) = \\ & \quad \Pr\left[\theta_j > t, \tau_i \geq t\right] e^{-rt} L(p_i(t+)) - \Pr\left[\theta_j > t_0, \tau_i \geq t_0\right] e^{-rt_0} L(p_i(t_0+)) \\ & \quad + \Pr\left[\theta_j = t, \tau_i \geq t\right] e^{-rt} M(\phi(t)) - \Pr\left[\theta_j = t_0, \tau_i \geq t_0\right] e^{-rt_0} M(\phi(t_0)) \\ & \quad + \mathbb{E}\left[e^{-r\theta_j} W\left(\phi(\theta_j)\right) \mathbf{1}_{\{t_0 \leq \theta_j < \min\{t, \tau_i\}\}}\right]. \end{aligned} \tag{6}$$

Notice that $u_i(t, \sigma_j)$ is continuous over any open interval $(\underline{t}, \bar{t})$ such that $\sigma_j(\{t\}) = 0$ for any $t \in (\underline{t}, \bar{t})$.

A.2 Proofs for Section 3

A.2.1 Proof of Theorem 1

The proof of Theorem 1 is organized in four steps. First, I prove that in any symmetric equilibrium, the distribution is non-atomic. Second, I show that the support of any equilibrium strategy must be an interval. Third, I characterize the unique equilibrium candidate. Last, I prove that the unique equilibrium candidate is indeed an equilibrium.

Non-atomicity

I start establishing a property of each player's best reply. Fix a strategy σ_j for player j.

Lemma 3. *Let $\sigma_j(\{t_0\}) > 0$, for some $t_0 \in \mathbb{R}_+$. If σ_i is a best-reply to σ_j, then $\sigma_i([t_0 - \varepsilon, t_0]) = 0$ for some $\varepsilon > 0$.*

Proof. I prove that $\lim_{\varepsilon \to 0} u_i(t_0 + \varepsilon, \sigma_j) > u_i(t_0, \sigma_j)$. The result then follows from the left continuity of $u_i(t, \sigma_j)$ on $(t_0, t_0 + \varepsilon)$ (for some sufficiently small ε). Since

$M(p) < F(p) \leq W(p)$ for any $p \in (0, 1)$,

$$\lim_{\varepsilon \to 0} u_i(t_0 + \varepsilon, \sigma_j) = \Pr[\theta_j > t_0, \tau_i \geq t_0] \, e^{-rt_0} L(p_i(t_0+)) + \mathbb{E}\left[e^{-rt_0} W(\phi(t_0)) \mathbf{1}_{\{\theta_j \leq \min\{t_0, \tau_i\}\}}\right]$$
$$> \Pr[\theta_j > t_0, \tau_i \geq t_0] \, e^{-rt_0} L(p_i(t_0+)) + \Pr[\theta_j = t_0, \tau_i \geq t_0] \, e^{-rt_0} M(\phi(t_0))$$
$$+ \mathbb{E}\left[e^{-r\theta_j} W(\phi(\theta_j)) \mathbf{1}_{\{\theta_j < \min\{t_0, \tau_i\}\}}\right] = u_i(t_0, \sigma_j).$$

Hence, the result follows. □

Lemma 3 remains valid in the absence of payoff externalities, i.e., when $L(\omega_i) = M(\omega_i) = F(\omega_i)$, for $\omega_i \in \{G, B\}$ but requires a different proof (see Rosenberg et al., 2013). In the absence of payoff externalities, the result follows from the fact that in a symmetric equilibrium, information cannot come in bursts. To the contrary, when payoff externalities are present, the best-reply property stated in Lemma 3 is a result of the tie-breaking rule, that is, of the assumption that $M(G) < F(G)$. In fact, if $M(G) = F(G) > L(G)$, there exist symmetric equilibria in which the equilibrium strategy has atoms, while Lemma 3 readily implies that any symmetric equilibrium distribution is non-atomic.

Corollary 1. *Let (σ, σ) be an equilibrium strategy profile. Then, σ is non-atomic.*

Interval support

I now argue that in any symmetric equilibrium, the support of the distribution is an interval with lower endpoint t_*. (In the following lemmas, σ_j is assumed to be nonatomic.)

Lemma 4. *Let (t_1, t_2) be such that $\sigma_j((t_1, t_2)) = 0$ and let $t_0 \in (t_1, t_2)$ be such that $p_i(t_0) > p_*$. Then, $t \mapsto u_i(t, \sigma_j)$ is decreasing over $[t_0, t_2)$.*

Proof. Since $\sigma_j((t_1, t_2)) = 0$, $p_i(t)$ is increasing over (t_1, t_2). I want to show that $u_i(t, \sigma_j) < u_i(t_0, \sigma_j)$ for each $t \in (t_0, t_2)$. Simplifying (6) gives

$$\frac{u_i(t, \sigma_j) - u_i(t_0, \sigma_j)}{e^{-rt_0} \Pr[\tau_i \geq t_0, \theta_j > t]} = \left(p_i(t_0) L(G) + (1 - p_i(t_0)) e^{-\lambda(t - t_0)} L(B)\right) e^{-r(t - t_0)} - L(p_i(t_0)).$$

Because $p_i(t) > p_*$, the right-hand side of this last equality is negative, so $u_i(t, \sigma_j) - u_i(t_0, \sigma_j) < 0$. □

Lemma 5. *Let $t_0 \in \mathbb{R}_+$ be such that $p_i(t_0+) < p_*$. Then, $t \mapsto u_i(t, \sigma_j)$ is increasing over the interval $[t_0, t_0 + \varepsilon]$ for $\varepsilon > 0$ small enough.*

Proof. First, notice that $p_i(t_0+) < p_*$ implies $p_i(t+) < p_*$ for all $t > t_0$ close enough to t_0. From (6), using the fact that $F(\omega_i) \geq M(\omega_i) \geq L(\omega_i)$, for any $\omega_i = B, G$,

$$u_i(t, \sigma_j) - u_i(t_0, \sigma_j) \geq \Pr[\theta_j \geq t, \tau_i \geq t] \, e^{-rt} L(p_i(t)) - \Pr[\theta_j > t_0, \tau_i \geq t_0] \, e^{-rt_0} L(p_i(t_0+))$$
$$+ \mathbb{E}\left[e^{-r\theta_j} W(\phi(\theta_j)) \mathbf{1}_{\{t_0 < \theta_j < \min\{t, \tau_i\}\}}\right].$$

Using the fact that

$$\mathbb{E}\left[e^{-r\theta_j} W\left(\phi(\theta_j)\right) \mathbf{1}_{\{t_0<\theta_j<\min\{t,\tau_i\}\}}\right] \geq \mathbb{E}\left[e^{-r\theta_j} W(\phi(\theta_j)) \mathbf{1}_{\{\theta_j\in(t_0,t),\tau_i\geq t\}}\right]$$
$$\geq \mathbb{E}\left[e^{-rt} L(\phi(\theta_j)) \mathbf{1}_{\{\theta_j\in(t_0,t),\tau_i\geq t\}}\right],$$

one has that

$$u_i(t,\sigma_j) - u_i(t_0,\sigma_j) \geq \Pr\left[\theta_j > t_0, \tau_i \geq t_0\right] e^{-rt_0}$$
$$\cdot \left(\left(p_i(t_0+)L(G) + (1-p_i(t_0+))e^{-\lambda(t-t_0)}L(B)\right) e^{-r(t-t_0)} - L(p_i(t_0+))\right).$$

Since $p_i(t_0+) < p^*$, the right-hand side of this last equality is positive so that $u_i(t,\sigma_j) - u_i(t_0,\sigma_j) > 0$. □

By Corollary 1 and Lemma 5, in any symmetric equilibrium, $p_i(t) \geq p_*$ for any t in the support of the equilibrium strategy, while by Lemma 4, the lower endpoint of the support of the equilibrium strategy is t_*.

Lemma 6. *Let (σ,σ) be a symmetric equilibrium. Then, the support of σ is an interval.*

Proof. Assume by contradiction that there exists t_1, $t_2 \in \operatorname{supp}\sigma$ such that $(t_1,t_2) \notin \operatorname{supp}\sigma$. Since σ is non-atomic, $u_i(t,\sigma)$ is continuous on $[t_1,t_2]$ and by Lemma 4 is decreasing in (t_1,t_2), which contradicts the indifference condition $u_i(t_1,\sigma) = u_i(t_2,\sigma)$. □

Equilibrium candidate

The next proposition provides necessary conditions for a strategy of player j, σ_j, to make player i indifferent between waiting and investing over an arbitrary interval of time. Let $G_j(t) := \sigma_j([0,t])$ and define $H_j(t) := \int_0^t e^{-\lambda x}(1-G_j(x))\,\mathrm{d}x$.

Proposition 6. *Let $[\underline{t},\overline{t}]$ be a non-empty interval such that $\sigma_j(\{t\}) = 0$ for any $t \in [\underline{t},\overline{t}]$. Then, the following two statements are equivalent:*

(i) the map $t \mapsto u_i(t,\sigma_j)$ is constant over $[\underline{t},\overline{t}]$;

(ii) on the interval $[\underline{t},\overline{t}]$, the function $H_j(t)$ is of class $\mathbf{C}^2$ and is a solution to the linear, second-order equation

$$(\phi(t)-p_*)\,H_j'(t) + \lambda H_j(t)\,(1-\phi(t))\left(\Pr[\omega_i = G \mid \omega_j = B, \tau_i \geq t] - p_*\right)$$
$$= -p_*\left(\frac{W\left(\phi(t)\right) - L\left(\phi(t)\right)}{-(\lambda+r)L(B)}\right)\left(H_j''(t) + \lambda H_j'(t)\right). \tag{7}$$

Proof. I first prove that the first statement implies the second. Using (6), the equality $u_i(t,\sigma_j) = u_i(t+\varepsilon,\sigma_j)$ writes

$$\Pr\left[\theta_j \geq t, \tau_i \geq t\right] L(p_i(t)) = \Pr\left[\theta_j \geq t+\varepsilon, \tau_i \geq t+\varepsilon\right] e^{-r\varepsilon} L(p_i(t+\varepsilon))$$
$$+\mathbb{E}\left[e^{-r(\theta_j-t)} W\left(\phi(\theta_j)\right) \mathbf{1}_{\theta_j<\min\{t+\varepsilon,\tau_i\}}\right].$$

Dividing by ε and taking the limit as $\varepsilon \to 0$,

$$rp_i(t)L(G)+(r+\lambda)(1-p_i(t))L(B) = (W(\phi(t)) - L(\phi(t))) \lim_{\varepsilon\to 0} \frac{\Pr[\theta_j < t+\varepsilon \mid \theta_j \geq t, \tau_i \geq t]}{\varepsilon}$$

Recall that $p_* := -(r+\lambda)L(B)/(rL(G)-(r+\lambda)L(B))$ is the optimal cutoff belief in the single-agent problem in which the net present value payoff from investing is $L(\omega)$. Dividing by $-(\lambda+r)L(B)$ and multiplying by p_*,

$$p_i(t) - p_* = p_* \left(\frac{W(\phi(t)) - L(\phi(t))}{-(\lambda+r)L(B)}\right) \lim_{\varepsilon\to 0} \frac{\Pr[\theta_j < t+\varepsilon \mid \theta_j \geq t, \tau_i \geq t]}{\varepsilon}. \tag{8}$$

Equation (8) implies that $G_j(t)$ must be a continuous differentiable function on the interval $[\underline{t}, \overline{t}]$. Hence,

$$\lim_{\varepsilon\to 0} \frac{\Pr[\theta_j \in [t, t+\varepsilon), \tau_i \geq t]}{\varepsilon} = \Pr[\tau_i \geq t, \tau_j \geq t] G_j'(t).$$

Also,

$$\begin{aligned}\Pr[\theta_j \geq t, \tau_i \geq t] = &\Pr[\tau_i \geq t, \tau_j \geq t](1-G_j(t)) \\ &+ \left(p_0(1-p_0)(1-\rho) + (1-p_0)(1-p_0+\rho p_0)e^{-\lambda t}\right)\lambda H_j(t),\end{aligned}$$

and

$$\begin{aligned}\Pr[\omega_i = G, \theta_j \geq t, \tau_i \geq t] = &p_0(p_0 + (1-p_0)\rho + (1-p_0)(1-\rho)e^{-\lambda t})(1-G_j(t)) \\ &+ p_0(1-p_0)(1-\rho)\lambda H_j(t).\end{aligned}$$

Plugging into (8), after a few manipulations,

$$\begin{aligned}(\phi(t) - p_*)(1-G_j(t)) + e^{\lambda t}\lambda H_j(t)(1-\phi(t))(\Pr[\omega_i = G \mid \omega_j = B, \tau_i \geq t] - p_*) \\ = p_* \left(\frac{W(\phi(t)) - L(\phi(t))}{-(\lambda+r)L(B)}\right) G_j'(t),\end{aligned}$$

where $\Pr[\omega_i = G \mid \omega_j = B, \tau_i \geq t]$ is a function of time only (see footnote 4). Replacing $1 - G_j(t)$ with $e^{\lambda t}H_j'(t)$, one obtains (7), as desired.

By these computations, if $H_j(t) := \int_0^t e^{-\lambda x}(1-G_j(x))\,\mathrm{d}x$ solves (7) on some interval, the map $t \mapsto u_i(t, \sigma_j)$ is differentiable on that interval with a derivative equal to zero. It follows that the second statement implies the first. □

Verification

It remains to show that there exists a symmetric strategy σ that satisfies the indifference condition in Proposition 6.

Notice that the coefficients of $H_j(t)$, $H'_j(t)$, and $H''_j(t)$ in (7) are continuous functions of $t \geq 0$ and are bounded away from 0 for any finite time t. Hence, given a pair of initial conditions, equation (7) has a unique solution on $[t_*, \infty)$.

Denote with $\hat{H}$ the unique solution to (7) on $[t_*, \infty)$ such that $\hat{H}(t_*) = (1 - e^{-\lambda t_*})/\lambda$, and $\hat{H}'(t_*) = e^{-\lambda t_*}$. Let $\bar{t} \coloneqq \inf\{t > t_* : e^{\lambda t}\hat{H}'(t) = 0\}$. Define the map $\hat{G} : \mathbb{R}_+ \to [0, 1]$ as $\hat{G}(t) = 0$ for $t \leq t_*$, $\hat{G}(t) = 1 - e^{\lambda t}H'(t)$ for $t \in [t_*, \bar{t}]$, and $\hat{G}(t) = 1$ for $t > \bar{t}$. The map $\hat{G}$ is continuously differentiable over $\mathbb{R}_+$. By (8) and the definition of t_*, $\hat{G}'(t_*) = 0$. Hence, by (1), $p'(t_*) > 0$ where $p : \mathrm{R}_+ \to \mathrm{R}$ is the unique solution to (1) such that $p(0) = p_0$.[11] This in turn implies by (8) that $G'(t) > 0$ for $t \in (t_*, \bar{t}]$.

Hence, G is the cdf of a non-atomic measure σ and p is equal to a player's belief when facing an opponent who plays σ. By Proposition 6, the map $t \mapsto u_i(t, \sigma)$ is constant on $[t_*, \bar{t}]$. Moreover, $p(t) < p_*$ for $t < t_*$, and $p(t) \geq p_*$ for $t > \bar{t}$. Thus, any strategy with support in $[t_*, \bar{t}]$ is a best-reply to σ.

A.2.2 Proof of Lemma 2

Consider first the case $\rho < 1$. Assume for the sake of contradiction that $\bar{t} = \infty$. Let rewrite (7) as

$$\kappa\, e^{\lambda t}\left(H''_j(t) + \lambda H'_j(t)\right) = -\frac{e^{\lambda t}\left(\phi(t) - p_*\right)}{W\left(\phi(t)\right) - L\left(\phi(t)\right)} H'_j(t) - \frac{e^{\lambda t}\left(1 - \phi(t)\right)\alpha(t)}{W\left(\phi(t)\right) - L\left(\phi(t)\right)} \lambda H_j(t), \tag{9}$$

with $\alpha(t) \coloneqq \Pr[\omega_i = G \mid \omega_j = B, \tau_i \geq t] - p_*$, and $\kappa > 0$. Define

$$t_\psi \coloneqq \inf\{t \in \mathbb{R}_+ : \phi(t) \geq p_* \text{ and } \Pr[\omega_i = G \mid \omega_j = B, \tau_i \geq t] \geq p_*\}.$$

It is easy to see that irrespective of $\rho < 1$, $t_\psi < \infty$. Let $\hat{t} > t_\psi$. Then, integrating,

$$\kappa e^{\lambda t} H'(t) = \kappa e^{\lambda \hat{t}} H'(\hat{t}) - \int_{\hat{t}}^{t} \frac{e^{\lambda s}\left(\phi(s) - p_*\right)}{W\left(\phi(s)\right) - L\left(\phi(s)\right)} H'(s)\, \mathrm{d}s - \lambda \int_{\hat{t}}^{t} \frac{e^{\lambda s}(1 - \phi(s))\alpha(s)}{W\left(\phi(s)\right) - L\left(\phi(s)\right)} H(s)\, \mathrm{d}s. \tag{10}$$

Using the definition of $\phi(t)$,

$$\lim_{t\to\infty} (1 - \phi(t))e^{\lambda t} = \frac{(1 - p_0)p_0(1 - \rho)}{p_0(p_0 + (1 - p_0)\rho)} > 0.$$

[11] More precisely, $p : \mathbb{R}_+ \to \mathbb{R}$ is implicitly defined by

$$\frac{p(t)}{1 - p(t)} = \begin{cases} \dfrac{p_0}{(1 - p_0)e^{-\lambda t}} & t < t_*, \\[2ex] \dfrac{p_0\left((p_0 + (1 - p_0)\rho)\, \hat{H}'(t)e^{\lambda t} + (1 - p_0)(1 - \rho)\left(\lambda \hat{H}(t) + \hat{H}'(t)\right)\right)}{(1 - p_0)e^{-\lambda t}\left((1 - p_0 + p_0\rho)\left(\lambda \hat{H}(t) + \hat{H}'(t)\right) + p_0\rho e^{\lambda t}\hat{H}(t)\right)} & t \geq t_*. \end{cases}$$

Recall that $W(\phi(t)) - L(\phi(t)) > 0$ for any $t \geq t_*$. It follows that the second integral in (10) converges to $-\infty$ and $e^{\lambda t} H'(t) \to -\infty$, which contradicts $\bar{t} = \infty$.

Now consider the case $\rho = 1$. The differential equation (7) simplifies to

$$\begin{aligned} &(\phi(t) - p_*)\, H'(t) - \lambda H(t)\,(1-\phi(t))\, p_* \\ &= -p_* \left(\frac{W(\phi(t)) - L(\phi(t))}{-(\lambda + r) L(B)} \right) \left(H''(t) + \lambda H'(t) \right). \end{aligned} \tag{11}$$

Assume by contradiction that $\bar{t} < \infty$. Hence, $H'(\bar{t}) = 0$ and $H'(t) > 0$ for $t < \bar{t}$. By (11) $H''(t) > 0$ for some neighborhood of $\bar{t}$, which contradicts $H'(t)$ crossing zero from above.

It is apparent from (9) that if $\rho < 1$, so that $\bar{t} < \infty$, $e^{-\lambda \bar{t}} \left(H_j''(\bar{t}) + \lambda H_j'(\bar{t}) \right) \neq 0$, which implies that the equilibrium distribution has a kink at $\bar{t}$.

A.3 Proofs for Section 4

A.3.1 Proof of Proposition 1

The proof is organized in three steps. First, I derive a differential equation for the belief $p_i(t)$ (rather than a differential system for $p_i(t)$ and $G(t)$). Second, I show that for different correlation parameters, the belief paths are ordered pointwise. Third, I show that the ranking of belief paths implies that of equilibrium distributions.

Differential equation for $p_i(t)$

Let σ_j be a given non-atomic strategy of player j and let $p_i(t)$ be the belief of player i when facing the strategy σ_j. As before, let $G_j(t) \coloneqq \sigma_j([0,t])$.

Lemma 7. *Let $[\underline{t}, \bar{t}] \subset \operatorname{supp} \sigma_j$. The following two statements are equivalent:*

(i) the map $t \mapsto u_i(t, \sigma_j)$ is constant over $[\underline{t}, \bar{t}]$;

(ii) on the interval $[\underline{t}, \bar{t}]$, the function $p_i(t)$ is of class $\mathbf{C}^1$ and is a solution to

$$p_i'(t) = \lambda\, p_i(t)(1 - p_i(t)) - (\phi(t) - p_i(t))\, \frac{rL(p_i(t)) + \lambda(1-p_i(t))L(B)}{W(\phi(t)) - L(\phi(t))} \tag{12}$$

Proof. I show that (8) is equivalent to (12). After rearranging, (8) is equivalent to

$$\frac{rL(p_i(t)) + \lambda(1-p_i(t))L(B)}{W(\phi(t)) - L(\phi(t))} = \Pr\left[\tau_j > t \mid \theta_j \geq t, \tau_i \geq t\right] \frac{G_j'(t)}{1 - G_j(t)}. \tag{13}$$

Since conditional on ω_i, τ_i is independent of θ_j and τ_j,

$$\frac{\Pr\left[\tau_j \geq t \mid \omega_i = G, \theta_j \geq t\right]}{\Pr\left[\tau_j > t \mid \theta_j \geq t, \tau_i \geq t\right]} = \frac{\phi(t)}{p_i(t)}, \text{ and } \frac{\Pr\left[\tau_j \geq t \mid \omega_i = B, \theta_j \geq t\right]}{\Pr\left[\tau_j > t \mid \theta_j \geq t, \tau_i \geq t\right]} = \frac{1-\phi(t)}{1-p_i(t)}.$$

Using these relationships and (13) to substitute for $G_j'(t)/\left(1 - G_j(t)\right)$ in (1) yields (12). $\square$

Ranking of the belief paths

Recall that $F(\omega) = L(\omega) + \Delta$, $\omega = G, B$. With this normalization, the function W can be written as

$$W(p) = \begin{cases} \left(\frac{p}{1-p}\,\frac{1-p^*}{p^*}\right)^{r/\lambda} \frac{p}{p^*}\left(p^* L(G) + (1-p^*)L(B) + \Delta\right) & \text{if } p \le p^*, \\ pL(G) + (1-p)L(B) + \Delta & \text{if } p > p^*, \end{cases}$$

Lemma 8. *If $\rho \ge 0$ ($\rho \le 0$), the equilibrium belief path $p_i^{\Delta}(t)$ increases (respectively decreases) in Δ for all $t \ge 0$.*

Proof. In equilibrium, players invest with positive probability only after t_*, and t_* does not change with Δ; hence, it is sufficient to prove that the ranking holds for all $t \ge t_*$.

Because $\phi(t) > p_i(t)$ if and only if $\rho > 0$, the right-hand side of (12) is increasing in Δ if and only if $\rho > 0$. On $[t_*, \bar{t}]$, $p_i(t)$ is the unique solution of the first-order differential equation (12) satisfying $p_i(t_*) = p_*$, where again p_* does not depend on Δ. The result then follows from a standard comparison argument (see Teschl, 2012, Theorem 1.3).

□

Ranking of the equilibrium distribution

Let $\Gamma : \mathbf{C}^1(\mathbb{R}_+) \to \mathbf{C}^2(\mathbb{R}_+)$ be the one-to-one map between belief paths $p_i(t)$ and distribution functions $G_j(t)$. The map $\Gamma : p \mapsto G$ is implicitly defined by

$$\begin{aligned} \frac{p}{1-p} &= \frac{p_0}{(1-p_0)\,e^{-\lambda t}} \\ &\cdot \frac{(p_0 + (1-p_0)\rho)\,(1-G(t)) + (1-p_0)\rho\left(e^{-\lambda t}\,(1-G(t)) + \lambda \int_0^t e^{-\lambda s}\,(1-G(s))\,\mathrm{d}s\right)}{(1-p_0+p_0\rho)\left(\lambda \int_0^t e^{-\lambda s}\,(1-G(s))\,\mathrm{d}s + e^{-\lambda t}\,(1-G(t))\right) + p_0\rho\,(1-G(t))}. \end{aligned} \tag{14}$$

Lemma 9. *If $\rho \ge 0$ ($\rho \le 0$), the map $\Gamma : \mathbf{C}^1(\mathbb{R}_+) \to \mathbf{C}^2(\mathbb{R}_+)$ is decreasing (increasing) over the set of functions $p \in \mathbf{C}^1(\mathbb{R}_+)$ such that $p(0) = p_0$.*

Proof. As is clear from rewriting (14) as

$$\frac{p(t)}{1-p(t)} = \frac{p_0}{(1-p_0)\,e^{-\lambda t}} \frac{(p_0 + (1-p_0)\rho)\,e^{\lambda t} + (1-p_0)\rho\left(1 + \frac{\lambda\int_0^t e^{-\lambda s}(1-G(s))\,\mathrm{d}s}{e^{-\lambda t}(1-G(t))}\right)}{(1-p_0+p_0\rho)\left(\frac{\lambda\int_0^t e^{-\lambda s}(1-G(s))\,\mathrm{d}s}{e^{-\lambda t}(1-G(t))} + 1\right) + p_0\rho e^{\lambda t}}, \tag{15}$$

for $p, q \in \mathbf{C}^1(\mathbb{R}_+)$, if $p(t) \ge q(t)$ for all $t \ge 0$ and $\rho > 0$, then

$$\frac{\int_0^t e^{-\lambda s}\,(1-\Gamma(p)(s))\,\mathrm{d}s}{e^{-\lambda t}\,(1-\Gamma(p)(t))} \le \frac{\int_0^t e^{-\lambda s}\,(1-\Gamma(q)(s))\,\mathrm{d}s}{e^{-\lambda t}\,(1-\Gamma(q)(t))},$$

while the opposite inequality holds for $\rho < 0$.

I now show that for $G^\dagger, G^\ddagger \in \mathbf{C}^2(\mathbb{R}_+)$, if $G^\dagger(0) = G^\ddagger(0) = 0$, and for all $t \geq 0$

$$\frac{\int_0^t e^{-\lambda s}\left(1 - G^\dagger(s)\right)\mathrm{d}s}{1 - G^\dagger(t)} \geq \frac{\int_0^t e^{-\lambda s}\left(1 - G^\ddagger(s)\right)\mathrm{d}s}{1 - G^\ddagger(t)},$$

then $G^\dagger(t) \geq G^\ddagger(t)$ for all $t \geq 0$.

Define $\bar{t} = \inf\{t : G^\dagger(t) < G^\ddagger(t)\}$ and assume by contradiction that $\bar{t}$ is finite. First, notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\int_0^t e^{-\lambda s}(1 - G(s))\,\mathrm{d}s}{1 - G(t)}\right) = e^{-\lambda t} + \frac{\int_0^t e^{-\lambda s}(1 - G(s))\,\mathrm{d}s}{1 - G(t)}\frac{G'(t)}{1 - G(t)}.$$

Hence, $\frac{\mathrm{d}G^\dagger(t)}{\mathrm{d}t}\Big|_{t=0} \geq \frac{\mathrm{d}G^\ddagger(t)}{\mathrm{d}t}\Big|_{t=0}$ and $\bar{t} > 0$. Since $G^\dagger$ and $G^\ddagger$ are continuous, $G^\dagger(\bar{t}) = G^\ddagger(\bar{t})$. Therefore,

$$\frac{\int_0^{\bar{t}} e^{-\lambda s}\left(1 - G^\dagger(s)\right)\mathrm{d}s}{1 - G^\dagger(\bar{t})} < \frac{\int_0^{\bar{t}} e^{-\lambda s}\left(1 - G^\ddagger(s)\right)\mathrm{d}s}{1 - G^\ddagger(\bar{t})},$$

which is a contradiction. □

Proposition 1 follows from combining Lemma 8 and Lemma 9.

A.3.2 Proof of Proposition 2

I divide the proof into two cases, depending on the sign of the correlation. For each case, given a symmetric strategy (σ, σ), I write the probability of investing in a bad project $\Pr[\theta_i < \infty \mid \omega_i = B]$ in terms of the distribution function $G : \mathbb{R}_+ \mapsto [0, 1]$, $G(t) := \sigma([0, t])$. I then show that $\Pr[\theta_i < \infty \mid \omega_i = B]$ is decreasing in G.

The result then follows from Proposition 1, which states that fixing all other parameters, the equilibrium strategies are ranked by first-order stochastic dominance.

For any distribution G, let $\hat{\theta}_j^G$ be the (improper) random variable with distribution function

$$t \mapsto ((1 - p_0) + p_0\rho)\left(1 - e^{-\lambda t}(1 - G(t)) - \int_0^t \lambda e^{-\lambda s}(1 - G(s))\,\mathrm{d}s\right) + (1 - p_0)(1 - \rho)G(t),$$

and let $\hat{\theta}_i^G$ be the random variable with distribution function $t \mapsto G(t)$.

Assume that both players invest according to the distribution G. Conditional on $\{\omega_i = B\}$, the time at which player i invests should he ignore player j's action has the same distribution as $\hat{\theta}_j^G$.

Given two distribution functions $G^\dagger : \mathbb{R}_+ \to [0, 1]$ and $G^\ddagger : \mathbb{R}_+ \to [0, 1]$ such that $G^\ddagger$ first-order stochastically dominates $G^\dagger$, i.e., $G^\ddagger(t) \leq G^\dagger(t)$, for all $t \geq 0$, the random variable $\hat{\theta}_j^{G^\ddagger}$ first-order stochastic dominates $\hat{\theta}_j^{G^\dagger}$. It follows that the random variable $\min\{\hat{\theta}_j^{G^\ddagger}, \hat{\theta}_i^{G^\ddagger}\}$ first-order stochastic dominates $\min\{\hat{\theta}_j^{G^\dagger}, \hat{\theta}_i^{G^\dagger}\}$, where for $G = G^\dagger, G^\ddagger$, $\hat{\theta}_j^G$ and $\hat{\theta}_i^G$ are understood to be independent random variables.

Positive correlation Assume that $G(t_*) = 0$, so that on path as soon as the first-mover invests, the second mover follows suit. Hence,

$$\begin{aligned}\Pr[\theta_i < \infty \mid \omega_i = B] &= \Pr\left[\tau_i \geq \hat{\theta}_j^G \mid \hat{\theta}_j^G < \hat{\theta}_i^G, \omega_i = B\right] + \Pr\left[\tau_i \geq \hat{\theta}_i^G \mid \hat{\theta}_j^G > \hat{\theta}_i^G, \omega_i = B\right] \\ &= \mathbb{E}\left[e^{-\lambda \min\{\hat{\theta}_j^G, \hat{\theta}_i^G\}}\right].\end{aligned}$$

In words, conditional on $\{\omega_i = B\}$, player i invests at some finite time if one of the following events occur. Either player j invests as a first mover, and by the time he does so, player i has not received any signal and follows suit. Alternatevely, player i invests as a first mover. The second equality is immediate since signals follow an exponential distribution. Since $t \mapsto e^{-\lambda t}$ is decreasing, the map from the set of distribution functions endowed with the first-order stochastic dominance order to the probability of investing in a bad project, $\Pr[\theta_i < \infty \mid \omega_i = B]$, is order-reversing.

Negative correlation Define $w(t) = \min\{s \geq 0 : \phi(t)/\left((1-\phi(t))e^{-\lambda s}\right) \geq p^*\}$. If the first mover invests at time t, the second mover optimally waits until time $\theta + w(\theta)$ before investing. Hence,

$$\begin{aligned}\Pr[\theta_i < \infty \mid \omega_i = B] = \Pr&\left[\tau_i \geq \hat{\theta}_j^G + w(\hat{\theta}_j^G) \mid \hat{\theta}_j^G < \hat{\theta}_i^G, \omega_i = B\right] \\ &+ \Pr\left[\tau_i \geq \hat{\theta}_i^G \mid \hat{\theta}_j^G > \hat{\theta}_i^G, \omega_i = B\right] \\ = \mathbb{E}&\left[e^{-\lambda \min\{\hat{\theta}_j + w(\hat{\theta}_j^G), \hat{\theta}_i\}}\right].\end{aligned}$$

Since the first-order stochastic dominance order is closed under increasing transformation, by the same reasoning, $\Pr[\theta_i < \infty \mid \omega_i = B]$, is order-reversing.

A.3.3 Proof of Proposition 3

Recall that p_* is the optimal cutoff belief in the single-agent problem in which the payoff from investing is $L(\omega)$.

First, I show that there exists a $\kappa \in (0, -L(B))$ such that whenever $F(\omega) = L(\omega) + \kappa$ for $\omega = G, B$, $\phi(\bar{t}_\kappa) > p^*_\kappa$, where $\bar{t}_\kappa$ is the upper endpoint of the equilibrium distribution and p^*_κ is the optimal cutoff belief in the single-agent problem in which the payoff from investing is $L(\omega) + \kappa$.

By Proposition 1, the upper end of the support is increasing in κ. Further, the second mover cutoff is decreasing in κ and is equal to 0 for $\kappa = -L(B)$. Because $\phi(0) > 0$ and $\phi(t)$ is strictly increasing in t, the first statement follows.

Second, keeping $L(\omega)$ fixed, the cutoff p^* is increasing in both $F(G) - L(G)$ and in $F(B) - L(B)$. Further, inspection of the proof of Proposition 1 reveals that G is strictly decreasing pointwise both in $F(G) - L(G)$ and in $F(B) - L(B)$. (Specifically, the right-hand side of (12) is increasing both in $F(G) - L(G)$ and in $F(B) - L(B)$.) It follows that if $F(\omega) > L(\omega) + \kappa$ for $\omega = G, B$, $\phi(\bar{t}_\kappa) > p^*_\kappa$.

A.3.4 Proof of Proposition 4

When $\rho = 1$, $p_i(t)$ and $\phi(t)$ solve the (autonomous) system

$$\begin{cases} p_i'(t) = \lambda\, p_i(t)(1-p_i(t)) - (\phi(t)-p_i(t))\, \frac{rL(p_i(t))+\lambda(1-p_i(t))L(B)}{\phi(t)(F(G)-L(G))+(1-\phi(t))(F(B)-L(B))} \\ \phi'(t) = 2\lambda\phi(t)\,(1-\phi(t))\,. \end{cases} \tag{16}$$

Since $\phi(t)$ converges to 1 as $t \to \infty$, for any $\varepsilon > 0$ and t sufficiently large,

$$p_i'(t) = \lambda\, p_i(t)(1-p_i(t)) - (1-p_i(t))\, \frac{rL(p_i(t))+\lambda(1-p_i(t))L(B)}{F(G)-L(G)} + o(\varepsilon). \tag{17}$$

The right-hand side of (17) is positive if and only if $p_i(t) \geq \overline{p}$, where

$$\overline{p} := \frac{-\,L(B)\,(\lambda + r)}{(r+\lambda)\,(L(G)-L(B)) - \lambda F(G)}.$$

If $\overline{p} \in (0,1]$, then $\overline{p} \geq p_*$. Also, $(\overline{p}, 1)$ is an asymptotically stable point of the differential system (16). It follows that $p_i(t)$ converges and

$$\lim_{t\to\infty} p_i(t) = \overline{p}.$$

If $\overline{p} \notin (0,1)$, then $p_i(t)$ is increasing in t for any $t \geq 0$ and

$$\lim_{t\to\infty} p_i(t) = 1.$$

The statement of the proposition follows from the fact that $\overline{p} \in (0,1)$ if and only if $v > \lambda$.

A.4 Proofs for Section 5

A.4.1 Discussion of the Parametric Assumptions

To understand the role of the assumptions, consider the hypothetical scenario in which player j is allowed to invest only after player i, so that by construction $I^* = \{i\}$.[12] In any equilibrium that survives the refinement, after player i has invested, player j behaves according to the optimal single-agent policy described in Lemma 1: if player i invests at t, player j either invests at $t + w(t)$, where

$$w(t) := \min\left\{ w \geq 0 : \frac{\phi(t)}{(1-\phi(t))\,e^{-\lambda w} + \phi(t)} \geq p^* \right\},$$

[12] Again, I allow player j to invest with no delay after player i.

or abstain from investing if $\tau_j < t + w(t)$. Taking the behavior of the follower into account, player i solves a single-agent optimal stopping problem: he chooses the optimal investment time to maximize

$$\begin{aligned}\mathcal{L}(t) \coloneqq\, & e^{-rt}\left(p_0 L(G) + (1-p_0)e^{-\lambda t}L(B)\right)\\ &+ p_0\left(p_0 + (1-p_0)\rho + (1-p_0)(1-\rho)e^{-\lambda(t+w(t))}\right)e^{-r(t+w(t))}\Delta^G\\ &+ (1-p_0)\,e^{-\lambda t}\left(\left(1-p_0+p_0\rho\right)e^{-\lambda(t+w(t))} + p_0(1-\rho)\right)e^{-r(t+w(t))}\Delta^B,\end{aligned}$$

where $\Delta^\omega = F(\omega) - L(\omega)$, for $\omega = G, B$.

The expected payoff from investing (as a leader) at some time t depends on the expected quality of both projects, on the probability that the opponent has observed a signal, and on calendar time. As a result, the leader's intertemporal considerations are not limited to the standard trade-off between learning and discounting. The later the leader invests, the more likely it is that an opponent with a bad project follow suits. However, the later the leader invests, the shorter the time the follower waits before investing if he does follow suit. As shown below, the assumptions (i) $\rho > 0$, (ii) $p_0/(1-p_0) < -F(B)/F(G)$, and (iii) $(r+\lambda)\left(L(B) + p_0(1-\rho)\left(\Delta^G + \Delta^B\right)\right) + 2(r+2\lambda)(1-p_0+p_0\rho)\Delta^B \le 0$ guarantee that the function $\mathcal{L} : \mathbf{R}_+ \to \mathbf{R}$ is single-peaked.

First, notice that $w : \mathbf{R}_+ \to \mathbf{R}_+$ is continuous and non-increasing; it is of class $\mathbf{C}^2$ except at $t_\phi \coloneqq \min\{t \ge 0 : \phi(t) \ge p^*\}$. For any $t > t_\phi$, $w(t) = 0$. For any $t < t_\phi$

$$\begin{aligned}w'(t) = &-1\\ &- \frac{\rho e^{-\lambda t}}{\left(p_0(1-\rho) + e^{-\lambda t}\left(1-p_0+p_0\rho\right)\right)\left(p_0 + (1-p_0)\rho + e^{-\lambda t}(1-p_0)(1-\rho)\right)}.\end{aligned} \tag{18}$$

and

$$\begin{aligned}w''(t) = &-\lambda\left(1-\rho\right)\rho e^{-\lambda t}\\ &\cdot \frac{\left(1-p_0\right)\left(1-p_0+p_0\rho\right)e^{-2\lambda t} - p_0\left(p_0 + (1-p_0)\rho\right)}{\left(p_0(1-\rho) + e^{-\lambda t}\left(1-p_0+p_0\rho\right)\right)^2\left(p_0 + (1-p_0)\rho + e^{-\lambda t}(1-p_0)(1-\rho)\right)^2}.\end{aligned}$$

Hence, $w'(t) < -1$ and $w''(t) < 0$ if and only if $\rho > 0$. Let $t^* \coloneqq \min\{t \ge 0 : \pi(t) \ge p^*\}$.

Lemma 10. *Under the assumptions (i)-(iii), the function $\mathcal{L} : \mathbf{R}_+ \to \mathbf{R}$ is single-peaked, $\arg\max_t \mathcal{L}(t) =: t^\dagger \in (t^*, t_*)$, and $\mathcal{L}'(t^\dagger) = 0$.*

Proof. First, by assumptions (i) and (ii), $0 < t_\phi < t^*$. Consider the derivative of $\mathcal{L}(t)$ with respect to t, for $t \neq t_\phi$,

$$\begin{aligned}\mathcal{L}'(t) = & -e^{-rt}\Big(rp_0 L(G) + (r+\lambda)(1-p_0)e^{-\lambda t}L(B)\Big)\\ & - p_0\Big(r\,(p_0+(1-p_0)\rho) + (r+\lambda)(1-p_0)(1-\rho)e^{-\lambda(t+w(t))}\Big)\,e^{-r(t+w(t))}(1+w'(t))\Delta^G\\ & - (1-p_0)\,e^{-\lambda t}\Big((r+\lambda)\,(1-p_0+p_0\rho)\,e^{-\lambda(t+w(t))} + rp_0(1-\rho)\Big)\,e^{-r(t+w(t))}(1+w'(t))\Delta^B\\ & - (1-p_0)\,e^{-\lambda t}\lambda\Big((1-p_0+p_0\rho)\,e^{-\lambda(t+w(t))} + p_0(1-\rho)\Big)\,e^{-r(t+w(t))}\Delta^B.\end{aligned}$$

If $t < t_\phi$, assumption (i) implies that $1 + w'(t) < 0$. Further,

$$(r+\lambda)L(B) + \lambda\Big((1-p_0+p_0\rho)\,e^{-\lambda(t+w(t))} + p_0(1-\rho)\Big)\,e^{-w(t)}\Delta^B < (r+\lambda)F(B) < 0.$$

So, $\mathcal{L}'(t) > 0$ for any $t < t_\phi < t^*$.

At any $t > t_\phi$, $w(t) = 0$ and

$$\begin{aligned}\mathcal{L}''(t) = & -r\mathcal{L}'(t) + (1-p_0)e^{-(r+\lambda)t}(r+\lambda)\left(L(B) + p_0(1-\rho)\Delta^G + p_0(1-\rho)\Delta^B\right)\\ & + 2\lambda(1-p_0)e^{-(r+2\lambda)t}(r+2\lambda)\,(1-p_0+p_0\rho)\,\Delta^B\\ < & -r\mathcal{L}'(t),\end{aligned}$$

where the last inequality follows from assumption (iii). Hence, if $\mathcal{L}'(t) = 0$ for some $t > t_\phi$, then $\mathcal{L}''(t) < 0$. Combining these observations, one gets that the function $\mathcal{L}(t)$ is single-peaked and $t^\dagger \in [t_\phi, \infty)$. Last, it can be verified that $\mathcal{L}(t^*) < 0$ and by assumption (ii) $\mathcal{L}(t^*) > 0$, so that $t^\dagger > t^*$ and $\mathcal{L}(t^\dagger) = 0$. □

A.4.2 Proof of Theorem 2

Here I prove that the cdf of the unique symmetric equilibrium coincides on $[t^\dagger, \bar{t}]$ with the unique solution to the integro-diffential equation

$$\begin{aligned}&\Bigg(\big(\Pr[\omega_i = G, \omega_j = G \mid \tau_i \geq t, \tau_j \geq t]\,r + \Pr[\omega_i = B, \omega_j = G \mid \tau_i \geq t, \tau_j \geq t]\,(r+\lambda)\big)\frac{\Delta^G}{rL(G)}\\ &\quad + \big(\Pr[\omega_i = B, \omega_j = B \mid \tau_i \geq t, \tau_j \geq t]\,(r+2\lambda)\\ &\qquad + \Pr[\omega_i = G, \omega_j = B \mid \tau_i \geq t, \tau_j \geq t]\,(r+\lambda)\big)\frac{(r+2\lambda)\Delta^B}{rL(G)} + \frac{\phi(t)-p_*}{1-p_*}\Bigg)e^{\lambda t}(1-G(t)) \qquad (19)\\ &\quad + \Bigg(\Pr[\omega_i = G, \omega_j = B \mid \tau_i \geq t, \tau_j \geq t] + \Pr[\omega_i = B, \omega_j = B \mid \tau_i \geq t, \tau_j \geq t]\frac{p_*}{1-p_*}\Bigg)\\ &\qquad \cdot\left(1 - e^{-\lambda t^\dagger} + \lambda\int_{t^\dagger}^{t} e^{-\lambda x}(1-G(s))\,\mathrm{d}s\right) = 0,\end{aligned}$$

such that $\breve{G}(t^\dagger) = 0$.

As in the proof of Theorem 1, I denote with $u_i(t, \sigma_j)$ the expected payoff induced by the strategy profile (t, σ_j).

The proof parallels the one of Theorem 1. The generalizations of Lemma 3 and Corollary 1 are straightforward and omitted: in any symmetric equilibrium, the strategy is nonatomic. I first prove that the support of any equilibrium strategy must be an interval. Then, I characterize the unique equilibrium candidate and prove that it is indeed an equilibrium.

Interval support

I now argue that in any symmetric equilibrium, the support of the distribution is an interval with lower endpoint $t^\dagger$. (In the following lemmas, σ_j is assumed to be nonatomic.)

Lemma 11. *Suppose* $(0, \underline{t}) \notin \operatorname{supp} \sigma_j$ *and* $\sigma_j([\underline{t}, \underline{t}')) > 0$ *for some* $\underline{t} < \underline{t}' < t^\dagger$*. Then,* $t \mapsto u_i(t, \sigma_j)$ *is increasing over the interval* $[\underline{t}, \underline{t} + \varepsilon)$ *for some* $\varepsilon > 0$*.*

Proof. For $\varepsilon > 0$ sufficiently small

$$u_i(\underline{t} + \varepsilon, \sigma_j) - u_i(\underline{t}, \sigma_j) \geq \mathcal{L}(\underline{t} + \varepsilon) - \mathcal{L}(\underline{t}) > 0.$$

□

Lemma 12. *Suppose* $\min \operatorname{supp} \sigma_j > t^\dagger$*. Then,* $t \mapsto u_i(t, \sigma_j)$ *is decreasing over the interval* $[t^\dagger, \min \operatorname{supp} \sigma_j]$*.*

Proof. For any t and t', $t^\dagger \leq t < t' \leq \min \operatorname{supp} \sigma_j$,

$$u_i(t', \sigma_j) - u_i(t, \sigma_j) = \mathcal{L}(t') - \mathcal{L}(t) > 0.$$

□

Lemma 11 and Lemma 12 imply that in any symmetric equilibrium σ, $\min \operatorname{supp} \sigma = t^\dagger$.

Lemma 13. *Suppose* $(t_1, t_2) \notin \operatorname{supp} \sigma_j$ *for some* $t^\dagger < t_1 < t_2$*. If* $u_i(t_1, \sigma_j) = u_i(t_2, \sigma_j)$*,* $u_i(t, \sigma_j) > u_i(t_1, \sigma_j)$ *for some* $t \in (t_1, t_2)$*.*

Proof. Fix $t \in (t_1, t_2)$. Differentiating $u_i(t, \sigma_j)$ with respect to time,

$$\begin{aligned}
&\frac{\partial u_i(t, \sigma_j)}{\partial t} \\
&= e^{-rt} \Pr[\tau_i \geq t, \theta_j \geq t_1] \\
&\quad \cdot \big(-rp(t)L(G) - (r + \lambda)(1 - p(t))L(B) \\
&\qquad - p(t) \left(r \Pr\left[\omega_j = G \mid \omega_i = G, \theta_j \geq t_1\right] + (r + \lambda) \Pr\left[\omega_j = B, \tau_j \geq t \mid \omega_i = G, \theta_j \geq t_1\right]\right) \Delta^G \\
&\qquad - (1 - p(t)) \left((r + 2\lambda) \Pr\left[\omega_j = B, \tau_j \geq t \mid \omega_i = B, \theta_j \geq t_1\right]\right. \\
&\qquad\qquad\qquad \left. + (r + \lambda) \Pr\left[\omega_j = G \mid \omega_i = B, \theta_j \geq t_1\right] \right) \Delta^B \big).
\end{aligned}$$

Differentiating further,

$$
\begin{aligned}
&\frac{\partial^2 u_i(t,\sigma_j)}{\partial t^2} \\
&\quad = -r\frac{\partial u_i(t,\sigma_j)}{\partial t} \\
&\qquad + e^{-rt}\Pr[\tau_i \geq t, \theta_j \geq t_1] \\
&\qquad\quad \cdot \lambda \left((r+\lambda)(1-p(t))L(B) + p(t)\left((r+\lambda)\Pr[\omega_j = B, \tau_j \geq t \mid \omega_i = G, \theta_j \geq t_1]\right)\Delta^G\right. \\
&\qquad\qquad + (1-p(t))\left((r+2\lambda)\Pr[\omega_j = B, \tau_j \geq t \mid \omega_i = B, \theta_j \geq t_1]\right. \\
&\qquad\qquad\qquad \left.\left. +(r+\lambda)\Pr[\omega_j = G \geq t \mid \omega_i = B, \theta_j \geq t_1]\right)\Delta^B\right).
\end{aligned}
$$

Since

$$
\Pr[\tau_i \geq t, \theta_j \geq t_1](1-p(t)) \geq (1-p_0)e^{-\lambda t}\Pr\left[\theta_j \geq t_1 \mid \tau_i \geq t, \tau_j \geq t_1\right]
$$

it holds that

$$
\begin{aligned}
&\Pr[\tau_i \geq t, \theta_j \geq t_1] \\
&\quad \cdot \big((r+\lambda)(1-p(t))L(B) + p(t)\left((r+\lambda)\Pr[\omega_j = B, \tau_j \geq t \mid \omega_i = G, \theta_j \geq t_1]\right)\Delta^G \\
&\qquad + (1-p(t))\left(2(r+2\lambda)\Pr[\omega_j = B, \tau_j \geq t \mid \omega_i = B, \theta_j \geq t_1]\right. \\
&\qquad\qquad \left. +(r+\lambda)\Pr[\omega_j = G, \tau_j \geq t \mid \omega_i = B, \theta_j \geq t_1]\right)\Delta^B\big) \\
&\leq (1-p_0)e^{-\lambda t}\big((r+\lambda)L(B) + p_0(1-\rho)(r+\lambda)\Delta^G + e^{-\lambda t}\left(1-p_0+p_0\rho\right)2(r+2\lambda)\Delta^B \\
&\qquad + p_0(1-\rho)(r+\lambda)\Delta^B\big)\Pr\left[\theta_j \geq t_1 \mid \tau_i \geq t, \tau_j \geq t_1\right] < 0.
\end{aligned}
$$

where the last inequality follows from assumption (iii). Hence, $u_i(t,\sigma_j)$ is strictly concave whenever weakly increasing. Since $t \mapsto u_i(t,\sigma_j)$ is continuously differentiable in (t_1,t_2), if $u_i(t_1,\sigma_j) = u_i(t_2,\sigma_j)$, it follows that $u_i(t,\sigma_j) > u_i(t_2,\sigma_j)$ for some $t \in (t_1,t_2)$. □

It follows from Lemma 13 that the support of any symmetric equilibrium distribution must be an interval.

Equilibrium candidate

The next proposition provides necessary conditions for a strategy of player j, σ_j, to make player i indifferent between waiting and investing over an arbitrary interval of time. Let $G_j(t) := \sigma_j([0,t])$ and define $H_j(t) := \int_0^t e^{-\lambda s}(1-G_j(s))\,\mathrm{d}s$.

Proposition 7. *Let $[\underline{t},\overline{t}]$ be a non-empty interval such that $\underline{t} > t^\dagger$ and $\sigma_j(\{t\}) = 0$ for any $t \in [\underline{t},\overline{t}]$. Then, the following two statements are equivalent:*

(i) the map $t \mapsto u_i(t,\sigma_j)$ is constant over $[\underline{t},\overline{t}]$;

(ii) on the interval $[\underline{t}, \overline{t}]$, the function $H_j(t)$ is of class $\mathbf{C}^2$ and is a solution to the linear, first-order equation

$$\begin{aligned}
&\big(\phi(t) r L(G) + (1-\phi(t))(r+\lambda)L(B) + r\Pr\left[\omega_i = G, \omega_j = G \mid \tau_i \geq t, \tau_j \geq t\right]\Delta^G \\
&\quad + (r+\lambda)\Pr\left[\omega_i = B, \omega_j = G \mid \tau_i \geq t, \tau_j \geq t\right]\Delta^G \\
&\quad + (r+2\lambda)\Pr\left[\omega_i = B, \omega_j = B \mid \tau_i \geq t, \tau_j \geq t\right]\Delta^B \\
&\quad + (r+\lambda)\Pr\left[\omega_i = G, \omega_j = B \mid \tau_i \geq t, \tau_j \geq t\right]\Delta^B\big) H_j'(t) \\
&+ \big(\Pr[\omega_i = G, \omega_j = B \mid \tau_i \geq t, \tau_j \geq t] r L(G) \\
&\quad + \Pr[\omega_i = B, \omega_j = B \mid \tau_i \geq t, \tau_j \geq t](r+\lambda)L(B)\big)\lambda H_j(t) = 0
\end{aligned} \tag{20}$$

Proof. I first prove that the first statement implies the second. The equality $u_i(t, \sigma_j) = u_i(t+\varepsilon, \sigma_j)$ for $t \geq t^\dagger$ writes

$$\begin{aligned}
&\Pr\left[\tau_i \geq t, \theta_j \geq t\right] \\
&\quad \cdot \big(p_i(t)\left(L(G) + \Pr[\tau_j \geq t \mid \omega_i = G, \theta_j \geq t]\Delta^G\right) \\
&\qquad + (1-p_i(t))\left(L(B) + \Pr[\tau_j \geq t \mid \omega_i = B, \theta_j \geq t]\Delta^B\right)\big) \\
&= \Pr\left[\tau_j \geq t+\varepsilon, \tau_i \geq t+\varepsilon\right] e^{-r\varepsilon} \\
&\quad \cdot \big(p_i(t+\varepsilon)\left(L(G) + \Pr[\tau_j \geq t+\varepsilon \mid \omega_i = G, \theta_j \geq t+\varepsilon]\Delta^G\right) \\
&\qquad + (1-p_i(t+\varepsilon))\left(L(B) + \Pr[\tau_j \geq t+\varepsilon \mid \omega_i = B, \theta_j \geq t+\varepsilon]\Delta^B\right)\big) \\
&\quad + \mathbb{E}\left[e^{-r(\theta_j - t)} W\left(\phi(\theta_j)\right)\mathbf{1}_{\theta_j < \min\{t+\varepsilon, \tau_i\}}\right].
\end{aligned}$$

Dividing by ε and taking the limit as $\varepsilon \to 0$,

$$\begin{aligned}
&rp(t)L(G) + (r+\lambda)(1-p(t))L(B) \\
&\quad + p(t)\left(r\Pr\left[\omega_j = G \mid \omega_i = G, \theta_j \geq t\right] + (r+\lambda)\Pr\left[\omega_j = B, \tau_j \geq t \mid \omega_i = G, \theta_j\right]\right)\Delta^G \\
&\quad + (1-p(t))\big((r+\lambda)\Pr\left[\omega_j = B, \tau_j \geq t \mid \omega_i = B, \theta_j \geq t\right] \\
&\qquad\qquad + r\Pr\left[\omega_j = G \mid \omega_i = B, \theta_j \geq t\right] + \lambda\Pr\left[\tau_j \geq t \mid \omega_i = B, \theta_j \geq t\right]\big)\Delta^B \\
&= \lim_{\varepsilon \to 0} \frac{\Pr[\theta_j < t+\varepsilon \mid \theta_j \geq t, \tau_i \geq t]}{\varepsilon} \\
&\quad \cdot \left(W\left(\phi(t)\right) - \left(\phi(t)\left(L(G) + \Delta^G\right) + (1-\phi(t))\left(L(B) + \Delta^B\right)\right)\right).
\end{aligned} \tag{21}$$

For $t \geq t^\dagger$, the right-hand side is equal to zero. Algebraic manipulations of (21) yield (20).

By these computations, if $H_j(t) := \int_0^t e^{-\lambda s}(1 - G_j(s))\,\mathrm{d}s$ solves the differential equation on some interval, the map $t \mapsto u_i(t, \sigma_j)$ is differentiable on that interval with a derivative equal to zero. It follows that the second statement implies the first. $\square$

Verification

First, let H be the unique solution to (20) on the interval $[t^\dagger, \infty)$ such that $H(t^\dagger) = (1 - e^{-\lambda t^\dagger})/\lambda$. Define $\overline{t} := \inf\{t : e^{\lambda t} H'(t) = 0\}$.

The differential equation (20) can be written as

$$H'(t) + \lambda H(t) = h(t)H'(t), \tag{22}$$

where $h : [t^\dagger, \infty] \to \mathbf{R}_+$ is a continuously differentiable function defined as

$$\begin{aligned} h(t) = -&\frac{1}{(1-p_0)p_0(1-\rho)e^{-\lambda t} r L(G) + (1-p_0)(1-p_0+p_0\rho)e^{-2\lambda t}(r+\lambda)L(B)} \\ &\cdot \big(p_0(p_0 + (1-p_0)\rho)r(L(G)+\Delta^G) + (1-p_0)(1-p_0+p_0\rho)e^{-2\lambda t}(r+2\lambda)\Delta^B \\ &\quad + (1-p_0)p_0(1-\rho)e^{-\lambda t}(r+\lambda)(L(B)+\Delta^B+\Delta^G)\big). \end{aligned}$$

Recall that $t_\psi \coloneqq \inf\{t \in \mathbb{R}_+ : \Pr[\omega_i = G \mid \omega_j = B, \tau_i \geq t] \geq p_*\}$. Notice that t_ψ is finite if and only if $\rho < 1$. For any $t \in [t^\dagger, t_\psi)$, $h(t) > 1$ and $\lim_{t \to t_\psi^-} h(t) = \infty$. Also, the function $H(t)$ is of class $\mathbf{C}^2$ in $[t^\dagger, t_\psi)$.

Lemma 14. *For any $t \in [t^\dagger, t_\psi)$, if $h(t) > 1$, $h'(t) > \lambda h(t)$.*

Proof. First, it can be shown that for any $t \in [t^\dagger, t_\psi)$, the denominator of $h(t)$ is negative. Also,

$$\begin{aligned} h'(t) =& \lambda h(t)\Bigg(- \Big(2(1-p_0)(1-p_0+p_0\rho)e^{-2\lambda t}(r+2\lambda)\Delta^B \\ &\qquad + (1-p_0)p_0(1-\rho)e^{-\lambda t}(r+\lambda)(L(B)+\Delta^B+\Delta^G)\Big) \\ &\quad \cdot \Big(p_0(p_0+(1-p_0)\rho)r(L(G)+\Delta^G) + (1-p_0)(1-p_0+p_0\rho)e^{-2\lambda t}(r+2\lambda)\Delta^B \\ &\qquad + (1-p_0)p_0(1-\rho)e^{-\lambda t}(r+\lambda)(L(B)+\Delta^B+\Delta^G)\Big)^{-1} \\ &\quad + \frac{(1-p_0)p_0(1-\rho)e^{-\lambda t} r L(G) + 2(1-p_0)(1-p_0+p_0\rho)e^{-2\lambda t}(r+\lambda)L(B)}{(1-p_0)p_0(1-\rho)e^{-\lambda t} r L(G) + (1-p_0)(1-p_0+p_0\rho)e^{-2\lambda t}(r+\lambda)L(B)}\Bigg) \end{aligned}$$

Using assumption (iii), one can show that

$$\begin{aligned} -2(1-p_0+p_0\rho)e^{-2\lambda t}(r+2\lambda)\Delta^B - p_0(1-\rho)e^{-\lambda t}(r+\lambda)(L(B)+\Delta^B+\Delta^G) \\ > (1-p_0+p_0\rho)e^{-2\lambda t}(r+\lambda)L(B). \end{aligned}$$

This inequality, together with the fact that $h(t) > 1$, implies that $h'(t) > \lambda h(t)$. □

Lemma 15. *For any $t \in [t^\dagger, t_\psi)$, if $H'(t) > 0$ and $h(t) > 1$, $H''(t) + \lambda H'(t) \leq 0$.*

Proof. From (22),

$$H''(t) + \lambda H'(t) = h(t)H''(t) + h'(t)H'(t) \geq h(t)\left(H''(t) + \lambda H'(t)\right),$$

which implies that $H''(t) + \lambda H'(t) < 0$. □

To complete the proof, notice that from boundary condition $H(t^\dagger) = (1-e^{-\lambda t^\dagger})/\lambda$ and the fact that $\mathcal{L}'(t^\dagger) = 0$, it follows that $H'(t^\dagger) = e^{-\lambda t^\dagger}$. It can be verified that $h(t^\dagger) > 1$ and $\lim_{t\to t_\psi} h(t) = \infty$. Hence, by Lemma 15, $H'(t)$ is decreasing in $[t^\dagger, t_\psi)$ and from (22), that $\bar{t} < t_\psi$. Let define the map $\hat{G} : \mathbb{R}_+ \to [0,1]$ as $\hat{G}(t) = 0$ for $t \le t^\dagger$, $\hat{G}(t) = 1 - e^{\lambda t} H'(t)$ for $t \in [t^\dagger, \bar{t}]$, and $\hat{G}(t) = 1$ for $t > \bar{t}$. The map $\hat{G}$ is continuously differentiable over $\mathbb{R}_+$,$G'(t) \ge 0$ for $t \in [t^\dagger, \bar{t}]$ and $\lim_{t\to\bar{t}} G(\bar{t}) = 1$. Hence, G is the cdf of a non-atomic measure σ. By Proposition 7, the map $t \mapsto u_i(t,\sigma)$ is constant on $[t^\dagger, \bar{t}]$. Thus, any strategy with support in $[t^\dagger, \bar{t}]$ is a best-reply to σ.

A.4.3 Proof of Proposition 5

Lemma 16. *Under assumption (i), (ii), and (iii), $t^\dagger$ is increasing in Δ.*

Proof. By Lemma 10, $\mathcal{L}'(t^\dagger) = 0$. Now,

$$\begin{aligned}\mathcal{L}'(t) = &- e^{-rt}\left(rp_0 F(G) + (r+\lambda)(1-p_0)e^{-\lambda t}\right)\\ &+ p_0(1-p_0)(1-\rho)\left(r(1-e^{-\lambda t}) - \lambda e^{-\lambda t}\right)\Delta\\ &+ (1-p_0)(1-p_0+p_0\rho)e^{-\lambda t}\left((r+\lambda)(1-e^{-\lambda t}) - \lambda e^{-\lambda t}\right)\Delta,\end{aligned}$$

where, by assumption (ii), the right hand side is increasing in Δ for any $t > t^*$. It follows that $t^\dagger$ is increasing in Δ. □

Given Δ, let σ^Δ be the equilibrium strategy and H^Δ be the unique solution of (20) over $[t^\dagger, \infty)$ satisfying $H^\Delta(t^\dagger) = (1-e^{-\lambda t})/\lambda$, where the dependence of $t^\dagger$ on Δ is omitted for notational convenience.

Because $t^\dagger$ is increasing in Δ, $H^{\overline{\Delta}}(t) > H^{\underline{\Delta}}(t)$ for $t \in (\min\operatorname{supp}\sigma^{\underline{\Delta}}, \min\operatorname{supp}\sigma^{\overline{\Delta}})$. I shall show that for $\overline{\Delta} > \underline{\Delta}$, the functions $H^{\overline{\Delta}}$ and $H^{\underline{\Delta}}$ cannot cross on $\operatorname{supp}\sigma^{\underline{\Delta}} \cap \operatorname{supp}\sigma^{\overline{\Delta}}$. Let $\tilde{t} \coloneqq \sup\{t \ge \min\operatorname{supp}\sigma^{\overline{\Delta}} : H^{\overline{\Delta}}(t) \ge H^{\underline{\Delta}}(t)\}$. Assume by contradiction that $\tilde{t} \in \operatorname{supp}\sigma^{\underline{\Delta}} \cap \operatorname{supp}\sigma^{\overline{\Delta}}$. In (22), $h(t)$ is increasing in Δ; hence, since $h(t) > 1$ in the relevant range, $H'^{\overline{\Delta}}(\tilde{t}) > H'^{\underline{\Delta}}(\tilde{t})$, which contradicts the definition of $\tilde{t}$.

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