Dynamic Signaling in Wald Options*

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Abstract

A sender engages in costly signaling to influence a decision maker, who observes a biased noisy signal and decides when to irreversibly take an action to match the binary state. We characterize Markov equilibria in terms of a two-dimensional boundary value problem for fixed discount rates and present sufficient conditions on the primitives for the two types of sender to choose an action higher than the myopically optimal action in all equilibria. We obtain a sharp characterization of equilibrium behavior when either the sender, the decision maker, or both become arbitrarily patient. The leading example is a dynamic limit pricing game between an incumbent and a potential entrant who uses the price to infer the industry conditions. A sufficiently patient incumbent always produces at capacity, and consumers can be hurt because the potential entrant strategically delays its entry.

Keywords: Dynamic Signaling, Optimal Stopping, Continuous-time, Wald Problem.

JEL Codes: C73, D21, D43, D82, D83

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1 Introduction

In light of the long-standing literature on real options (e.g. Dixit and Pindyck, 1994 and Stokey, 2009), the tradeoff between delay and more accurate information faced by a decision maker who has the option to take an irreversible action is well understood. However, in strategic settings, interested parties may be able to affect the signal the decision maker acquires before taking action. For example, venture capital firms wish to invest only in successful projects, but startups can try to affect their own periodic performance reports. Similarly, workers have private information about their ability and can affect their performance, while an employer wants to promote only capable employees. In the same vein, a firm considering entering a market learns about its profitability by observing the prevailing price. According to the limit pricing paradigm (see, for example, Chapter 9.4 in Tirole, 1988), an incumbent firm has incentives to put downward pressure on the price to persuade the potential entrant that the demand is weak and deter its entry.

To analyze these situations, we study a dynamic game in continuous time between two long-lived players, a decision maker (DM) and a sender. As in Wald (1945), the DM observes a public signal about a binary payoff-relevant state of the world and acts when sufficiently convinced of one state. A sender, who is privately informed about the state of the world, and has state-independent preferences over the DM's action, can engage in costly effort to affect the public signal observed by the DM. We model the public signal as a diffusion process whose drift depends on the true state of the world and the sender's action. Following Orlov, Skrzypacz, and Zryumov (2020), we call this class of games strategic Wald option games.

The contribution of the paper is twofold. First, we provide a continuous-time framework to study dynamic signaling incentives: we obtain a tractable characterization of Markov equilibria of Wald option games without the need to focus on linear equilibria, common in dynamic signaling models with Gaussian information structure. Our characterization complements the traditional adverse selection approach to reputation, providing a framework to conduct policy and welfare analysis, because the equilibrium predictions do not depend on the endogenously specified behavioral types. Second, we leverage the equilibrium characterization to revisit and get new insights into the dynamics of limit pricing. The dynamic limit pricing model illustrates the analytical traction of the continuous-time setup, but the strategic Wald option game framework has far-reaching applications. Our results could be applied, for example, to study the signaling dynamics of drug approval, as in Henry and Ottaviani (2019); or, in political economy, to study lobbying by interest groups or efforts to influence the public opinion when an agent (e.g., parliament, incumbent government) decides the moment at which consultations must stop and a decision has to be reached, in the spirit of Brocas and Carrillo (2007) and Salas (2019).

As mentioned above, we consider as our leading example a dynamic model of limit pricing à la Milgrom and Roberts (1982) and Matthews and Mirman (1983). The binary underlying uncertainty captures the state of the demand, which can be strong or weak. The sender is a privately informed incumbent firm that chooses its output to affect the market price, which is given by a linear demand curve perturbed by Brownian noise. The DM is a potential entrant who decides when if ever, to pay an entry cost to become an incumbent's competitor or take an outside option.

The sender has state-independent preferences in that it always prefers the potential entrant to take the outside option, regardless of the state of the demand. Affecting the public signal, that is, putting pressure on the price, involves producing a quantity other than the monopoly quantity, foregoing short-term profits.

We study equilibria that are Markov in the public belief about the state of the world. We provide a characterization of Markov equilibria for fixed discount rates in terms of a system of non-linear second-order ordinary differential equations. In equilibrium, both the value functions of the two types of sender and their actions are determined by a solution to a boundary value problem. A key step of our proof of equilibrium existence is showing that this multidimensional boundary value problem has a bounded solution, as there is no general existence theory for such problems. This technical result, of independent interest, is based on the method of upper and lower solutions.

It is intuitive, and we verify that it is true, that in any equilibrium of the limit pricing example, both types of incumbent have an incentive to put downward pressure on the price. In general, however, players' incentives to choose an action higher or lower than the myopically optimal action depend on the sensitivity of the continuation value to public signals, an equilibrium object. We also provide sufficient conditions in terms of the primitives of the game to guarantee that in any equilibrium both types of sender choose an action higher than the myopically optimal action.

We leverage our characterization to investigate equilibrium outcomes as the players become arbitrarily patient. First, for a fixed level of patience of the DM, if the sender's discount rate is sufficiently low, both types of sender find it optimal to choose the same external action at any belief, foregoing short-run gains.

Second, we fix the sender's discount rate and look at the limit as the DM becomes arbitrarily patient. In the limit, the cutoffs at which the DM acts shift closer to the extreme values (i.e., 0 and 1): the DM acts only when the uncertainty has vanished and takes a perfectly informed action. As a result, in equilibrium, both types of sender forfeit manipulating the DM's belief and choose the myopically optimal action at any point on the equilibrium path, because persuading the DM to take the sender's favorite action is too costly for an impatient sender.

Third, we consider the case when both discount rates converge to zero at the same speed. As in the previous case, in the limit, the DM acts only at extreme beliefs; however, unlike before, the cost of manipulation does not increase unboundedly. Because the sender is becoming patient at the same speed as the DM, even if persuading the DM takes more time, in the limit, both types of sender choose the same extremal action.

In our leading application, we investigate the dynamic welfare cost of limit pricing. First, as pointed out by Sweeting et al. (2020), the stylized two-period models that dominate the theoretical industrial organization literature are not suited to deliver empirical prediction on observed price patterns when a potential entrant can wait for several years before entering. Our model offers a complementary explanation to the patterns documented by Sweeting et al. (2020) in the airline industry, who report that incumbents not only cut prices when Southwest first appears as a potential entrant but keep prices low even when entry does not occur for quite long periods of time. Second, while the trade off between lower prices and entry delay is frequently mentioned by practitioners,[1] the existing literature has focused on the welfare costs due to inefficient entry, but, to the best of our knowledge, has overlooked the intensive time margin.

In our model, on the one hand, if the incumbent adopts an aggressive strategy consumers will be better off because of the price cuts. On the other hand, depending

[1]For example, in the recent Intel antitrust case, the EU Commission cited "a direct and immediate negative impact on those customers who would have had a wider price and quality choice," while Intel argued that "price declines brought large gains to the ultimate consumers who purchase computers." (Case COMP/C-3/37.990–Intel, Commission Decision, 2009, OJ C 227, 13–17).

on the effect of this aggressive pricing on the informativeness of the signal observed by the potential entrant, entry may be delayed, ultimately hurting consumers. We use our results to shed light on the efficacy of output restriction rules, which have been proposed for example by Williamson (1977) and Edlin (2002) as an antitrust policy tool to mitigate predatory behavior.

As a second application, we consider a promotion game between an employer (the DM) and an agent (the sender) who is privately informed about his ability. The agent's stochastic performance is affected by his ability and his choice of effort. The employer chooses whether and when to promote the agent, and would like to promote only the capable agent. The agent would like to be promoted regardless of his ability and, before promotion, is rewarded according to a pay-for-performance compensation scheme. In equilibrium, both types exert more effort than the myopically optimal level because the promotion decision is tied to the observed performance.

We show that, from an ex ante point of view, a "naive" sender can achieve a higher payoff than a strategic one. Specifically, a sufficiently patient agent who "commits" to the (type-dependent) myopically optimal action receives a higher expected payoff compared to the equilibrium. The idea is that when the agent exerts more effort than the myopically optimal level, the signal becomes more informative, leading the employer to adopt a more stringent promotion criterion, that, in turn, reduces the probability of promotion ultimately hurting the worker.

1.1 Related Literature

The paper belongs to the growing literature on dynamic signaling games with stopping decisions.[2] In Daley and Green (2012), Kolb (2015), Kolb (2019), Dilmé (2019), and Gryglewicz and Kolb (2022), the informed player, unlike in our paper, takes the stopping decision; further, in all these papers but Dilmé (2019), the informed player cannot directly manipulate the signal. Information manipulation is costless in the dynamic persuasion game of Orlov, Skrzypacz, and Zryumov (2020) when players' option exercise times are misaligned. Tangentially related, Henry and Ottaviani

[2] Bonatti, Cisternas, and Toikka (2017) and Cetemen (2021) study two-sided signaling models in continuous time, and Cisternas and Kolb (2024) analyze signaling in the presence of private monitoring. In contrast to our paper, these papers typically focus on linear Markov equilibria, Gaussian information structures, and finite horizon.

(2019) and McClellan (2022) analyze different versions of Wald persuasion games allowing the DM to commit.

At the intersection between the literature on dynamic signaling and the reputation literature, Ekmekci, Gorno, Maestri, Sun, and Wei (2022) study signal manipulation incentives in a dynamic principal-agent model in the presence of a commitment type.[3] Similarly to Pei (2021), and in contrast to the standard reputation models, we investigate reputation-building behavior as an equilibrium phenomenon. This difference also sets our paper apart from the reputation literature with long-run players, such as Cripps and Thomas (1997), Celetani et al. (1996) and Atakan and Ekmekci (2012, 2015).

The seminal work by Faingold and Sannikov (2011) studies reputation dynamics in a continuous-time game between a population of small players and a long-lived player, who can be a "normal" (i.e., strategic) type or a "commitment" type;[4] in contrast, we consider a Bayesian game with two long-run strategic players. Strategic types introduce significant complications, as both pooling and separating incentives are present and need to be considered in the equilibrium characterization. Technically, as compared to Faingold and Sannikov (2011), not only the equilibrium behavior of the sender is now characterized by a system of ODEs (instead of a single ODE), but also the presence of a long-run player taking a stopping decision introduces a fix point problem, absent in their paper.[5]

In our leading application, we analyze a continuous-time dynamic version of a limit pricing à la Milgrom and Roberts (1982) and Matthews and Mirman (1983). The latter is closer to our paper, because it allows for the possibility that the incumbent's first-period price is observed by the entrant with some noise. The vast majority of the empirical and theoretical literature builds upon the seminal contribution of Milgrom and Roberts (1982), whose framework is effectively static and not suitable for analyzing the effects of entry delay. We show how taking delay into account changes the welfare properties of the equilibrium. Saloner (1984) and Toxvaerd (2017)

[3] See also Ekmekci and Maestri (2022) for a discrete-time analogue of the model considered in Ekmekci et al. (2022).

[4] Bohren (2024) shows how their results extend to a more general class of stochastic games.

[5] Anderson and Smith (2013) and Dilmé (2024) rely on the tractability of continuous-time techniques to study a game between a long-run player and a sequence of short-run players to investigate. In both papers, the Gaussian signal only depends on the informed player's actions while we allow it to depend both on his action and his type.

extend the two-period model to multiple periods.[6] Unlike these papers, we do not endogenously impose a finite end to the game. Recently, Gryglewicz and Kolb (2023) study entry deterrence in a stopping game in which the incumbent, as in Milgrom and Roberts (1982), has private information about its costs and can choose to imitate a committed strong type via its pricing strategy.

2 Model

A sender and a decision maker (DM) interact over time. Time is continuous and potentially infinite, $t \in [0, \infty)$. A persistent state of the world θ determines the payoffs. The sender knows the state of the world $\theta \in \{H, L\} \subset \mathbf{R}$. At each time $t \geq 0$, the sender chooses an action $a_t \in A \subset \mathbf{R}$ from a compact interval, where $A := [\underline{a}, \bar{a}]$.

The DM decides when to take an irreversible action; that is, he chooses a stopping time τ, together with an action $b_\tau \in \{h, l\}$ to take at that time. The DM is uninformed about the state but observes at each point in time a signal which evolves according to

$$\mathrm{d}X_t = \mu(\theta, a_t)\,\mathrm{d}t + \sigma\,\mathrm{d}Z_t, \qquad X_0 = 0,$$

where $\sigma^2 > 0$, and Z_t is a standard Brownian motion which is independent of θ. We assume that $\mu : \{H, L\} \times A \to \mathbf{R}$ is Lipschitz continuous and non-increasing in its second argument, the action of the sender, and for $a \in A$, $\mu(H, a) \neq \mu(L, a)$. That is, the two types are statistically distinguishable when they take the same action. Nevertheless, there may exist a pair of feasible actions $a', a'' \in A$ that make the two types statistically indistinguishable, that is, $\mu(H, a') = \mu(L, a'')$.[7]

At each time t before the DM acts, the sender receives a flow payoff $\pi(\theta, a) \geq \underline{\pi}$, for some $\underline{\pi} \leq 0$. We assume that for any θ, and any a, $\pi(\theta, a)$ is a Lipschitz continuous and strictly concave function of the action. Let $a^*_\theta := \arg\max_a \pi(\theta, a)$ denote the

[6] Sweeting, Roberts, and Gedge (2020) build a finite-horizon analytically tractable model of dynamic limit pricing to structurally investigate the reduced-form evidence from Goolsbee and Syverson (2008).

[7] It is worth noting that, in contrast to the vast majority of papers studying continuous-time games in a Gaussian environment such as Cisternas (2018) and Bonatti et al. (2017), we do not impose an additive separable structure on the drift function.

myopic optimal action for the sender of type θ. We denote by $\pi^*(\theta) \coloneqq \pi(\theta, a^*_\theta)$ the myopic optimal payoff.

The DM and the sender discount the future at a rate $r_{DM} > 0$ and $r_S > 0$, respectively. If the DM acts at τ, his realized payoff is

$$\begin{cases} e^{-r_{DM}\tau} G(\theta, h) & \text{if } b_\tau = h, \\ e^{-r_{DM}\tau} G(\theta, l) & \text{if } b_\tau = l. \end{cases}$$

We assume that $G(H,h) > G(H,l)$ and $G(L,l) > G(L,h)$, and $\min\{G(H,h), G(L,l)\} > 0$ so the DM always wants to match the state.[8] At the time the DM acts, the sender collects a lump-sum payoff $\Pi(\theta, b_\tau)$, which depends on the state and the DM's terminal action. The sender always prefers the DM to choose action l, $\pi^*(\theta) = \Pi(\theta, l) > \Pi(\theta, h) \geq \underline{\pi}$. The assumption $\Pi(\theta, l) = \pi^*(\theta)$ captures the idea that once the sender obtains his favorite action, signaling concerns disappear, and he can achieve his myopic payoff in the (unmodelled) continuation game.[9]

A public strategy for the sender is a square-integrable process $(a_t)_{t\geq 0}$ that is progressively measurable with respect to the filtration generated by $(\theta, (X_t)_{t\geq 0})$. A strategy for the DM specifies a stopping time and an action to take when stopping that are progressively measurable with respect to the filtration generated by $(X_t)_{t\geq 0}$.

We denote by $(\phi_t)_{t\geq 0}$ the process of posterior belief that the DM attaches to $\theta = H$, where $(\phi_t)_{t\geq 0}$ is a progressively measurable process with respect to the filtration generated by $(X_t)_{t\geq 0}$, taking values in $[0,1]$. Hence, given a (public) strategy profile for the two types of sender, $(a_{t,H}, a_{t,L})$, by Liptser and Shiryaev (2001), the belief evolves according to

$$\begin{aligned} \mathrm{d}\phi_t = {} & \frac{\phi_t(1-\phi_t)\left(\mu(H, a_{t,H}) - \mu(L, a_{t,L})\right)}{\sigma} \\ & \cdot \frac{\mathrm{d}X_t - \left(\phi_t \mu(H, a_{t,H}) + (1-\phi_t)\,\mu(L, a_{t,L})\right)\mathrm{d}t}{\sigma}. \end{aligned} \tag{1}$$

The innovation process on the second line is a standard Brownian motion from the point of view of the DM. We define the speed of learning $\gamma(a_H, a_L, \phi)$ as the inverse

[8]In Section 5 and Section 6, we show that the model can be generalized to the case when the DM can take only one action but may want to prefer never to act.

[9]In Section 6 we show how the assumption can be relaxed to $\Pi(\theta, l) > \pi^*(\theta)$.

of the volatility of the DM's belief, that is,

$$\gamma(a_H, a_L, \phi) := \frac{\phi(1-\phi)\left(\mu(H, a_H) - \mu(L, a_L)\right)}{\sigma}.$$

It is determined by the DM's expectation about the action of each type of sender and the current belief ϕ, and it converges to 0 as ϕ approaches 0 or 1. Along the path of play, when contemplating a deviation, the sender anticipates that he cannot directly affect the speed of learning, because this speed is based on the DM's conjecture rather than on the actual action of the sender. The instantaneous choice of the sender can affect only the inference that the DM draws from the public signal by affecting the actual drift of the public signal. Higher $\gamma(a_H, a_L, \phi)$ implies that the belief ϕ_t reacts more to the public signal.

Given a strategy profile, the expected discounted payoff of the type θ sender can be written as

$$\mathbf{E}_\theta \left[\int_0^\tau r_S e^{-r_S s} \pi(\theta, a_s)\, \mathrm{d}s + e^{-r_S \tau} \left(\Pi(\theta, h) \mathbf{1}_{b_\tau = h} + \Pi(\theta, l) \mathbf{1}_{b_\tau = l} \right) \right].$$

Similarly, the expected discounted payoff of the DM given a strategy profile can be written as

$$\mathbf{E}\Bigg[e^{-r_{DM}\tau} \Bigg(\mathbf{1}_{b_\tau = h} \left(\phi_\tau G(H, h) + (1-\phi_\tau)\, G(L, h) \right) + \mathbf{1}_{b_\tau = l} \left(\phi_\tau G(H, l) + (1-\phi_\tau)\, G(L, l) \right) \Bigg) \Bigg].$$

We focus on equilibria that are Markovian in the posterior belief ϕ_t. A strategy profile for the sender is Markovian in ϕ_t if $(a_{t,H}, a_{t,L}) = (a_H(\phi_t), a_L(\phi_t))$ for some measurable function $a_\theta : [0,1] \to A \times A$. A strategy for the DM is Markovian in ϕ_t if $\tau = \inf\{t : \phi_t \notin \mathcal{D} \subset [0,1]\}$ a.s., and $b_\tau \in \arg\max\{\phi_\tau G(H,h) + (1-\phi_\tau)\, G(L,h), \phi_\tau G(H,l) + (1-\phi_\tau)\, G(L,l)\}$. Without loss of generality, we can assume that $\mathcal{D} = [\underline{\phi}, \overline{\phi}]$.

A Markov strategy profile together with a belief process $(\phi_t)_{t \geq 0}$ is a pure-strategy Markov equilibrium if at any time, along any public history,

(i) the DM's strategy solves his optimal stopping problem given the sender's strategy;

(ii) the sender's strategy maximizes his expected continuation payoff at any $\phi \in (\underline{\phi}, \bar{\phi})$;

(iii) the belief process $(\phi_t)_{t\geq 0}$ evolves according to (1) for $(a_{t,H}, a_{t,L}) = (a_H(\phi_t), a_L(\phi_t))$, given the initial prior ϕ_0.

3 Equilibrium Characterization

To prove the existence of and characterize Markov equilibria, we analyze the best-reply problem of the sender and DM, in turn.

Faingold and Sannikov (2011) prove existence by suggesting an appropriate continuous-time equivalent of the identifiability condition in Cripps et al. (2004): they assume that when the (strategic type of the) sender behaves myopically, his behavior is statistically distinguishable from the behavior of the commitment type. We show that in the absence of commitment types, it is enough to require that when different types of sender behave myopically optimally, they are statistically distinguishable. Specifically, we impose the following condition:

Condition 1. $\mu(H, a^*_H) \neq \mu(L, a^*_L)$.

Intuitively, for the two types to play a pair of observationally equivalent actions, the sender must be given intertemporal incentives. This requirement implies that in equilibrium, the volatility of the public belief is bounded above zero for any interior belief level.

3.1 Sender's Best-Reply Problem

Fix a Markov strategy for the DM together with a conjectured strategy profile for the sender $\hat{a}_\theta : [0, 1] \to A \times A$ used by the DM. Let $\underline{\phi}$ and $\bar{\phi}$ be the cutoff beliefs characterizing the DM's strategy, $\underline{\phi} < \bar{\phi}$; that is, the DM takes action h as soon as his posterior belief exceeds $\bar{\phi}$, and takes action l as soon as his posterior belief falls below $\underline{\phi}$. Given the DM's conjecture of the sender's actions, the sender faces a stochastic optimal control problem, because his action affects the drift of the belief process ϕ_t.

By using standard techniques, assuming that the value function $U_H : (0, 1) \to \mathbf{R}$ is twice continuously differentiable, we can write the Hamilton-Jacobi-Bellman (HJB) equation for the problem of the type H sender as

$$
\begin{aligned}
r_S U_H(\phi) = & \max_{a \in A} \left\{ r_S \pi(H,a) + \gamma\left(\hat{a}_H(\phi), \hat{a}_L(\phi), \phi\right) \frac{\mu(H,a)}{\sigma} U_H'(\phi) \right\} \\
& - \gamma\left(\hat{a}_H(\phi), \hat{a}_L(\phi), \phi\right) \frac{\phi \mu(H, \hat{a}_H(\phi)) + (1-\phi)\mu(L, \hat{a}_L(\phi))}{\sigma} U_H'(\phi) \\
& + \frac{1}{2} U_H''(\phi) \left(\gamma\left(\hat{a}_H(\phi), \hat{a}_L(\phi), \phi\right)\right)^2 .
\end{aligned}
\tag{2}
$$

When best replying, the sender trades off instantaneous payoffs (the first term in the parenthesis) and the effect that the sender's action has on the continuation payoff (the second term in the parenthesis). In a Markov equilibrium, the expected impact of today's action on the continuation payoff depends on its effect on the belief. The expected drift of the belief, from the point of view of the sender, is proportional to the drift of the public signal, which his action affects directly, and to the speed of learning. In turn, the sensitivity of the continuation payoffs to the belief is captured by the derivative of the value function (the last term in the parenthesis).

For each ϕ, given a conjectured strategy profile $(\hat{a}_H(\phi), \hat{a}_L(\phi))$ used by the DM and given the derivative of the value function $U_H'(\phi)$, (2) determines the optimal action for the type H sender. Similarly, the optimal action for the sender of type L is determined by

$$
\max_{a \in A} \left\{ r_S \pi(L,a) + \gamma\left(\hat{a}_H(\phi), \hat{a}_L(\phi), \phi\right) \frac{\mu(L,a)}{\sigma} U_L'(\phi) \right\} .
$$

In any Markov equilibrium, the DM's conjectured strategy profile is correct, and for any $\phi \in (\underline{\phi}, \bar{\phi})$, $(a_H(\phi_t), a_L(\phi_t)) = (a_H, a_L)$ solves the following system:

$$
\begin{aligned}
a_H &\in \operatorname*{argmax}_{a' \in A} \pi(H,a') + \frac{\mu(H,a_H) - \mu(L,a_L)}{\sigma} \frac{\mu(H,a')}{\sigma} z_H, \\
a_L &\in \operatorname*{argmax}_{a' \in A} \pi(L,a') + \frac{\mu(H,a_H) - \mu(L,a_L)}{\sigma} \frac{\mu(L,a')}{\sigma} z_L,
\end{aligned}
\tag{3}
$$

for $z_\theta = \phi(1-\phi)U_\theta'(\phi)/r_S$, $\theta \in \{H, L\}$. Intuitively, for any $(\phi, z_H, z_L) \in [0,1] \times \mathbf{R} \times \mathbf{R}$, any solution to the system identifies a Bayes Nash equilibrium of an auxiliary (one-shot) signaling game in which the sender is of type H with probability ϕ, and flow payoffs are perturbed by a "continuation-game term" weighted by (z_H, z_L). We require the following regularity conditions on this equilibrium correspondence $\mathcal{N}$: $(\phi, z_H, z_L) \mapsto (a_H, a_L)$.

Condition 2. *$\mathcal{N}(\phi, z^H, z^L)$ is a non-empty single-valued correspondence for each $(\phi, z^H, z^L) \in [0,1] \times \mathbf{R} \times \mathbf{R}$. Moreover, $\mathcal{N}$ is continuous on every bounded subset of $[0,1] \times \mathbf{R} \times \mathbf{R}$.*

In the Appendix, we show that Condition 2 is satisfied for the leading dynamic limit pricing application in Section 5 and for the promotion application in Section 6.

Condition 2 may seem at odds with the pervasiveness of equilibrium multiplicity in signaling games. However, even when the auxiliary one-shot signaling game has multiple equilibria, Theorem 1 characterizes equilibria satisfying a chosen static equilibrium selection (e.g., the Riley outcome) as long as the selector is continuous. Condition 2 implies that there exists $(\underline{z}_H, \underline{z}_L) \in \mathbf{R}_- \times \mathbf{R}_-$ such that for any $\phi \in [0,1]$, $\mathcal{N}(\phi, z^H, z^L) \in \{(\underline{a}, \underline{a}), (\bar{a}, \bar{a})\}$ whenever $(z^H, z^L) \in (-\infty, \underline{z}_H] \times (-\infty, \underline{z}_L]$. That is, if the signaling incentives are strong enough, in the unique equilibrium of the auxiliary one-shot signaling game, both types of sender play the same extremal action (see Lemma OA.2 in the Online Appendix).

Leveraging these observations, we state the following proposition, which provides a characterization of the sender's value functions in any Markov equilibrium. To put it differently, it characterizes the sender's pseudo-best reply (i.e., a mapping from the DM's strategy to the action profile for the two types of sender). As explained at the beginning of the section, in principle, when best replying, the sender must also take into account the conjecture used by the DM. We call these functions pseudo-best reply, rather than best reply, because in constructing them, we impose that the DM's conjecture is correct.

Proposition 1. *Assume Condition 1 and Condition 2 are satisfied. If the DM's strategy $\bar{\phi}$ and $\underline{\phi}$ and the sender's strategy profile $(a_H(\phi), a_L(\phi))$ are part of a Markov equilibrium, then the value functions of the sender solve the following system of second-order ordinary differential equations over the interval $(\underline{\phi}, \bar{\phi})$,*

$$
\begin{aligned}
U_H''(\phi) &= -2\frac{U_H'(\phi)}{\phi} + \frac{2r_S\left(U_H(\phi) - \pi(H, a_H(\phi))\right)}{\left(\gamma\left(a_H(\phi), a_L(\phi), \phi\right)\right)^2}, \\
U_L''(\phi) &= 2\frac{U_L'(\phi)}{1-\phi} + \frac{2r_S\left(U_L(\phi) - \pi(L, a_L(\phi))\right)}{\left(\gamma\left(a_H(\phi), a_L(\phi), \phi\right)\right)^2},
\end{aligned}
\tag{4}
$$

subject to the boundary conditions $U_\theta(\bar{\phi}) = \Pi(\theta, H)$*,* $U_\theta(\underline{\phi}) = \Pi(\theta, L)$*, and* $(a_H(\phi), a_L(\phi)) = \mathcal{N}(\phi, \phi(1-\phi)U_H'(\phi)/r_S, \phi(1-\phi)U_L'(\phi)/r_S)$ *for any* $\phi \in [\underline{\phi}, \bar{\phi}]$*. Moreover, for any* $0 < \underline{\phi} < \bar{\phi} < 1$*, the boundary value problem has a solution.*

There is no off-the-shelf result that guarantees the existence of a bounded solution to such a multidimensional second-order non-linear boundary value problem. Our proof of existence, of independent interest, is based on the method of upper and lower solutions and leverages the monotonicity of the system of differential equations.

Note that the system characterizes the sender's behavior only in the interval $\phi \in [\underline{\phi}, \bar{\phi}]$. We defer the discussion of the behavior of the sender outside the interval $[\underline{\phi}, \bar{\phi}]$ to Section 3.3.

3.2 DM's Best-Reply Problem

The best-reply problem of the DM is reminiscent of a Wald problem, as formulated by Chernoff (1972) and Moscarini and Smith (2003). That is, the DM engages in sequential testing of two hypotheses on the mean of a Wiener process. The following proposition characterizes the DM's best-reply problem.

In characterizing the best reply of the DM, we restrict attention to a regular strategy profile of the sender. The restriction is without loss of generality: it follows from Lemma 3 in the Appendix that in any sender's pseudo-best reply is regular, as defined next, because the coefficient γ is bounded away from zero.

Definition 1. *We call a strategy profile* $(a_H, a_L) : [0,1] \to A^2$ *regular if* $\gamma(a_H(\phi), a_L(\phi), \phi))^2 > 0$ *for all* $\phi \in (0,1)$ *and for all* $0 < \underline{\phi} < \bar{\phi} < 1$*,*

$$\int_{\underline{\phi}}^{\bar{\phi}} \frac{\sigma^2}{\gamma(a_H(\phi), a_L(\phi), \phi))^2} \,\mathrm{d}\phi < \infty, \qquad \sup_{\phi \in [\underline{\phi}, \bar{\phi}]} \frac{\gamma(a_H(\phi), a_L(\phi), \phi))^2}{\sigma^2} < \infty.$$

Regularity ensures that the stochastic differential equation governing the belief process has a weak solution that is unique in the sense of probability law, for all initial conditions. (See Section 5.5.C in Karatzas and Shreve, 1996.)

For any regular strategy profile for the sender, the solution to the optimal stopping problem faced by the DM can be characterized using standard techniques.

Proposition 2. *Fix a regular Markov strategy profile for the sender. The best reply of the DM is characterized by a pair of cutoffs together with a value function that* *solves in the interval* $[\underline{\phi}, \bar{\phi}]$

$$r_{DM}V(\phi) = \frac{1}{2}\left(\gamma(a_H(\phi), a_L(\phi), \phi)\right)^2 V''(\phi)$$

and at the boundaries satisfies

$$\begin{aligned} V(\underline{\phi}) &= \underline{\phi}G(H,l) + \left(1-\underline{\phi}\right)G(L,l), & V'(\underline{\phi}) &= G(H,l) - G(L,l), \\ V(\bar{\phi}) &= \bar{\phi}G(H,h) + \left(1-\underline{\phi}\right)G(L,h), & V'(\bar{\phi}) &= G(H,h) - G(L,h). \end{aligned}$$

Further, for any regular Markov strategy profile of the sender, there exists a unique optimal pair of cutoffs.

In characterizing the DM's best-reply problem, we verify that smooth-pasting holds for an arbitrary action profile of the sender as long as the difference in conditional variances is bounded away from zero, which guarantees that the stochastic differential equation governing the belief process has a unique weak solution.

3.3 Equilibrium Existence

Theorem 1. *Assume Condition 1 and Condition 2 are satisfied. Then, a Markov equilibrium exists.*

The existence of a Markov equilibrium follows from a fixed-point argument that combines Proposition 1 and Proposition 2. The only caveat in constructing an equilibrium is the behavior of the sender off the equilibrium path. Given the DM's strategy $\bar{\phi}$ and $\underline{\phi}$, Proposition 1 characterizes the behavior of the sender on the equilibrium path, that is, for beliefs $\phi \in [\underline{\phi}, \bar{\phi}]$. (Note that if the prior $\phi_0 > \bar{\phi}$, the DM acts immediately at and chooses h; if the prior $\phi_0 < \underline{\phi}$, the DM immediately chooses action l.) In proving the existence of a Markov equilibrium, we must complete the strategy profile outside the interval $[\underline{\phi}, \bar{\phi}]$.

A natural specification of the behavior off the equilibrium path is to assume that both types of sender choose the myopically optimal action at any $\phi \notin [\underline{\phi}, \bar{\phi}]$. In the proof of Theorem 1, we prove the existence of an equilibrium by requiring that for $\phi \notin [\underline{\phi}, \bar{\phi}]$, $a_\theta(\phi) = a^*_\theta$.

Intuitively, in a discrete-time approximation of our game, the sender's action at time t affects the belief of the DM at time $t + \Delta$. As the time between periods shrinks, starting from a history off the equilibrium path, with high probability, the belief will not leave the set $[0, \underline{\phi}) \cup (\bar{\phi}, 1]$ regardless of the action of the sender. In discrete-time dynamic games, sequential rationality would then imply that off the path, the sender plays the myopically optimal action. One way of adapting this notion to continuous time was formalized by Kuvalekar and Lipnowski (2020), who suggested an instantaneous sequential rationality refinement.

On the one hand, the refinement has some bite in that it is possible to construct spurious equilibria in which the DM is induced to act as soon as the belief enters some region in the anticipation that the two types of sender will adopt a strategy that makes the difference in conditional drift nil. For example, if for any $\phi \in [0, \phi_0 - \varepsilon) \cup (\phi_0 + \varepsilon, 1]$, $\mu(H, a_H(\phi)) = \mu(L, a_L(\phi))$, for some $\varepsilon > 0$ sufficiently small, the DM finds it optimal to act as soon as the belief leaves the interval $[\phi_0 - \varepsilon, \phi_0 + \varepsilon]$.[10] On the other hand, because the DM's best-reply problem satisfies a smooth pasting condition, the equilibria we construct could be sustained by specifying an alternative off-path behavior.[11]

On the one hand, Condition 1 guarantees that, in equilibrium, the variance of the belief process never vanishes as it does in Ekmekci et al. (2022). On the other hand, even if Condition 1 fails and $\mu(H, a_H^*) \neq \mu(L, a_L^*)$, the game still has a Markov equilibrium which is characterized by (4), but at one (or both) of the two cutoffs, the equilibrium behavior of the two types of the sender may converge to the myopic play so that the posterior no longer updates.

We conclude the section with a standard square root law of substitution between the discount rate and the volatility of the signals. We shall refer to this result in Section 4.2 when we study the limit as the players get patient.

Corollary 1. *Multiplying the discount rate of both the sender (r_S) and the DM (r_{DM}) by a factor of $\alpha > 0$ has the same effect on the equilibrium values and equilibrium behavior as rescaling the volatility σ by a factor of $\sqrt{\alpha}$.*

[10]Technically, the best-reply problem in Proposition 2 is well-defined only when the difference in conditional variance is bounded away from zero, but a limit argument can be provided to formally justify the claim.

[11]Ekmekci et al. (2022) circumvent the need to specify the off-path belief by introducing friction in the stopping problem, that is, the agent who takes the irreversible stopping decision can do so only upon receiving an opportunity that arrives according to a Poisson process.

The corollary immediately follows from the observation that the discount rates and the volatility parameter enter the DM's and the sender's problems only through the products $r_{DM}\sigma^2$ and $r_S\sigma^2$, respectively.

4 Equilibrium Properties

Throughout the paper, we assume Condition 1 and Condition 2 are satisfied. We now turn our focus to the general properties of the equilibria and conduct comparative statics with respect to the discount rate.

4.1 Signaling Incentives

Proposition 3. *In any Markov perfect equilibrium, the value functions of both types of sender are weakly decreasing, i.e.,* $U'_\theta(\phi) \leq 0$ *for all* $\phi \in [\underline{\phi}, \overline{\phi}]$.

Intuitively, the value functions are decreasing as both types of sender benefit from the DM holding a lower belief, that is, attaching a lower probability to $\theta = H$. The sender has incentives to deviate from his myopically optimal action to manipulate the public signal and induce a lower belief. However, whether a higher action increases or decreases the DM's belief depends on the sign of γ, which is pinned down by the conjectured equilibrium behavior.

Because the sender always prefers the DM to take action l as compared to action h, the type H of the sender always wants to pool with the type L of the sender, while the latter, in turn, wants to separate. These strategic considerations, absent in a model with a "commitment" type, can lead to two types of equilibrium, depending on the sign of γ. When γ is positive, in equilibrium, a lower public signal is interpreted as evidence in favor of $\theta = L$. As a result, the type H of the sender tries to put downward pressure on the signal by choosing an action higher than the myopically optimal action. At the same time, by assumption, if the type L of the sender were to choose the same action as the type H, he would induce an even lower signal, on average; in equilibrium, his effort to separate also translates in an action which is also higher than his myopically optimal action. The opposite dynamics ensues when γ is negative. We show in the appendix that γ never changes sign in equilibrium.

In the dynamic limit pricing application in Section 5, the difference in conditional drifts is always positive without the need for additional assumptions, and, as expected, both firms have incentives to put downward pressure on the price. In the following proposition, we provide sufficient conditions on the primitives for γ to be always positive.

Proposition 4. *Suppose that either condition (i) or (ii) below holds.*

(i) *(a)* $\Pi(H,l) - \Pi(H,h) > \Pi(L,l) - \Pi(L,h)$,

(b) $a^*_H > a^*_L$ *and for all* $x \in [a^*_L - \underline{a}, 0]$, $\pi_a(L, a^*_L - x) \geq \pi_a(H, a^*_H - x)$.

(c) $\mu_a(H,a) \leq \mu_a(L,a)$ *and* $\mu_{aa}(\theta,a) \leq 0$ *for each* $\theta \in \{H,L\}$ *and* $a \in A$.

(d) $\mu(H,a^*_H) - \mu(L,a^*_L) > 0$,

(ii) $\mu_a(\theta,a) \leq 0$ *for all* $a \in A$ *and each* $\theta \in \{H,L\}$, *and* $\mu(H,a^*_H) - \mu(L,\underline{a}) > 0$.

Then, in any equilibrium, for all $\phi \in [\underline{\phi}, \bar{\phi}]$, *both types of sender choose an action higher than their myopically optimal action, i.e.,* $a_\theta(\phi) \geq a^*_\theta$, *and an unexpectedly higher signal increases the public belief, i.e.,* $\gamma(a_H(\phi), a_L(\phi), \phi) > 0$.

Part (i) guarantees that not only type H has stronger incentives to induce action l (part (a)), but also that it is cheaper (part (b)) and easier (part (c)) for type H, as compared to type L, to put downward pressure on the signal. We show that with assumptions (a)-(c) part (d) suffices to guarantee that in any equilibrium, both types choose an action higher than the myopically optimal one. Conditions (a)-(c) are easy to satisfy, for example, when $\mu(\theta,a)$ is linear or separable in its argument.

Part (ii) can be understood as an identifiability assumption, strengthening Condition 1.

4.2 Patience Limits

Using our equilibrium characterization, we investigate signaling incentives and equilibrium outcomes as the players become arbitrarily patient. We need the following assumption, which allows us to derive a uniform lower bound on the speed of learning (see Lemma 4 in the Appendix).

Condition 3. *It holds that* $\mu(H,a^*_H) \neq \mu(L,a)$ *and* $\mu(L,a^*_L) \neq \mu(H,a)$ *for* $a \in \{\underline{a}, \bar{a}\}$.

Condition 3 strengthens the identifiability condition (Condition 1) and guarantees that in the limit as the players become arbitrary patient, the variance of belief does not vanish and the extreme form of ratchet effect which emerges in Ekmekci et al. (2022) does not occur.

Theorem 2. *Assume Condition 1, Condition 2, and Condition 3.*

1. *Fix the discount rate of the DM, $r_{DM} > 0$. If the sender is sufficiently patient, that is, for r_S low enough, in the unique Markov equilibrium, both types of sender play the same extremal feasible actions at any belief. If $\mu(H, a) - \mu(L, a) > 0$, then $a_H(\phi) = a_L(\phi) = \bar{a}$ for all $\phi \in [\underline{\phi}, \bar{\phi}]$; if $\mu(H, a) - \mu(L, a) < 0$, then $a_H(\phi) = a_L(\phi) = \underline{a}$ for all $\phi \in [\underline{\phi}, \bar{\phi}]$.*

2. *Fix the discount rate of the sender, $r_S > 0$. In the limit, as the DM becomes arbitrarily patient, $r_{DM} \to 0$, the strategy profile of both types of sender converges pointwise to the myopically optimal action, i.e., $a_H(\phi) \to a^*_H$ and $a_L(\phi) \to a^*_L$ for all $\phi \in [\underline{\phi}, \bar{\phi}]$. Moreover, $\underline{\phi} \to 0$ and $\bar{\phi} \to 1$.*

3. *Let $\{r_{S,n}\}_{n=1,2,...}$ and $\{r_{DM,n}\}_{n=1,2,...}$ be two sequences converging to zero such that $\lim_{n\to\infty} r_{S,n}/r_{DM,n} \to k \in (0, \infty)$. Then along any sequence of Markov equilibria, the strategy profile of both types of sender converges pointwise to the same extremal feasible action, i.e., either $a_\theta(\phi) \to \bar{a}$ or $a_\theta(\phi) \to \underline{a}$ for all $\phi \in [\underline{\phi}, \bar{\phi}]$, and $\theta \in \{H, L\}$, depending the sign of $\mu(H, a) - \mu(L, a)$, as in (1). Moreover, $\underline{\phi} \to 0$ and $\bar{\phi} \to 1$.*

The first two results in Theorem 2 can be understood in light of the tradeoff faced by the sender. First, as the sender becomes arbitrarily patient, his incentives to manipulate the DM's belief are stronger because short-term considerations become less salient; see (2). Notice that in our model, when both types choose the same extremal action, there is still information revelation in that the difference in expected conditional drifts is not zero.

Second, if the DM adopts an extreme strategy, that is, in the limit as the cutoff approaches 0 and 1, it is too costly for an impatient sender to try to engage in signaling, because such a strategy would have to involve a long period of belief manipulation. As the DM becomes arbitrarily patient, the marginal cost of waiting for additional information decreases, and the equilibrium cutoffs converge to 0 and 1. As a result, in the limit, neither type of sender engages in signaling and the equilibrium action profile converges to the myopically optimal action.

The third result combines the first two. To understand the intuition, consider a variation of our model in which the sender can only choose his action at time zero and cannot revise it at $t > 0$. For simplicity, assume that the sender can be of either types with equal probability. In this case, the best-reply problem of type $\theta \neq \vartheta$ of the sender can be written as

$$\max_{a \in A} \left(1 - \mathbf{E}[e^{-r_S \tau}]\right) \pi(\theta, a) + \mathbf{E}[e^{-r_S \tau}] \Bigg(\left(1 - \Phi\left(\frac{\mu(\theta, a) - \mu(\vartheta, a_\vartheta)}{2\sigma}\right)\right) \Pi(\theta, h) + \Phi\left(\frac{\mu(\theta, a) - \mu(\vartheta, a_\vartheta)}{2\sigma}\right) \Pi(\theta, l) \Bigg).$$

The cutoff beliefs used by the DM do not appear directly in the equation above,[12] but they determine the distribution of the stopping time τ. In the proof, we show that in the joint limit as both the sender and the DM become patient at comparable rates, optimality of the DM's behavior implies that the expected discount factor $\mathbf{E}[e^{-r_S \tau}]$ converges to 1. Inspection of the equation above reveals that the limit equilibrium must involve an extremal action for both types. While in the actual game, the sender has the ability to revise his action at any $t \in (0, \tau]$, in the limit, any gain from any such a revision is of a lower order.

In light of Corollary 1, the joint limit is equivalent to the limiting case as $\sigma \to 0$, perfect monitoring. In the proof, we also show that in the limit, the value function of the DM converges to his full-information value function. The expected time before the DM acts diverges as the cost of delay vanishes.

5 Dynamic Limit Pricing

As our leading application, we consider a dynamic model of entry deterrence à la Matthews and Mirman (1983). The sender is an incumbent firm that has private information about the state of the demand: it knows the demand shifter θ.[13] We interpret the action of the sender as its choice of output level, which results in a stochastic inverse demand,

$$\mathrm{d}X_t = \theta \left(1 - b \cdot a_t\right) \mathrm{d}t + \sigma \, \mathrm{d}Z_t, \qquad X_0 = 0,$$

[12]The DM takes action h (action l) whenever $\phi_\tau \geq \phi_0$ ($\phi_\tau \leq \phi_0$), and by assumption $\phi_0 = 1/2$.

[13]We conjecture that our equilibrium construction extends to the case when the demand fluctuates over time, as in Keller and Rady (1999), provided that the incumbent can observe the prevailing state.

where $\sigma^2 > 0$ is the variance, $b > 0$, and Z_t is a standard Brownian motion that is independent of θ. We assume that the set of feasible actions $A = [0, \bar{a}]$, for some $\bar{a} > 0$, which can be interpreted both as a capacity constraint or as an output restriction rule, as in Williamson (1977) and Edlin (2002). We will discuss the policy implication of this interpretation in Section 5.3.1.

The DM is a potential entrant who is uninformed about the state of the demand and decides when to take an irreversible action: becoming a competitor by entering the incumbent's market or taking an outside option. Heuristically, the potential entrant observes at each point in time the prevailing price $\mathrm{d}X_t/\,\mathrm{d}t$, a linear demand perturbed by an additive i.i.d. noise.

In line with this interpretation, at each time t, before the potential entrant acts, given the incumbent's output choice and the realization of the inverse demand, the resulting increment in the incumbent's profit is

$$\mathrm{d}X_t a_t - (c/2)a_t^2\,\mathrm{d}t, \qquad c > 0.$$

That is, for tractability, we assume that production costs are quadratic. We are interested in studying entry deterrence behavior rather than potential collusion in the contestable industry. Hence, we leave the continuation game after the potential entrant enters or takes the outside option unmodelled and attach continuation payoffs to capture the idea that the potential entrant would like to enter only if the demand is strong, $\theta = H$, while the incumbent is better off when its monopoly is unchallenged.

That is, if the potential entrant acts at τ, its payoff is

$$\begin{cases} e^{-r_E\tau}o & \text{if } b_\tau = l \\ e^{-r_E\tau}(D_\theta - F) & \text{if } b_\tau = h \end{cases},$$

where $D_H - F > o > D_L - F$, and until Section 5.3.1 $o > 0$. The incumbent's payoff in the game is

$$r_I \int_0^\tau e^{-r_I t} a_t\,\mathrm{d}X_t - r_I \int_0^\tau e^{-r_I t}(c/2)(a_t)^2\,\mathrm{d}t + e^{-r_I\tau}\left(M_\theta \mathbf{1}_{b_\tau=l} + D_\theta \mathbf{1}_{b_\tau=h}\right),$$

where $M_\theta > D_\theta$, for $\theta \in \{H, L\}$, $M_H > M_L$, and $D_H > D_L$. We set $M_\theta = \theta^2/(4b\theta+2c)$ and $D_\theta = (2b\theta + c)\theta^2/(2(3b\theta + c)^2)$, consistent with the assumption that firms are symmetric in their marginal cost and play a Cournot-Nash equilibrium in the duopoly continuation game with a known state.[14]

It may be argued that upon entry, the competitor is likely to not have access to the same information about the demand as the incumbent; or that even if the potential entrant takes the outside option, in the future, the incumbent's monopoly may be threatened by another potential competitor. In principle, it is possible to capture alternative information structures or the threat of future entry in the specification of the expected discounted payoffs that the firms collect at τ. While both the analytical and the numerical results we derive rely on the specific payoff assumptions, we believe that the insights generalize to these variations of the baseline model.

5.1 Equilibrium Properties

The following parametric assumptions are sufficient to guarantee that the baseline assumptions, as well as Condition 1, Condition 2, and Condition 3 hold in the dynamic limit pricing game.

Assumption 1. *The following conditions hold:*

(i) $H/(c+2bH) < \bar{a} < (H-L)/(bH) + H/L \cdot L/(c+2bL)$,

(ii) $(b(H-L) - bL + c)H > cL$.

Proving that these parametric restrictions are sufficient to guarantee Condition 2 requires a straightforward but tedious analysis of the pseudo-best reply. As shown in the Online Appendix, the pseudo-best reply of the low type is not necessarily single-valued, but these parametric restrictions guarantee that the equilibrium is unique. Assumption 1 fails, for example, when there is little demand uncertainty, that is, when the difference in demand intercepts is small. Besides ensuring non-negative flow profits, part (i) also implies that $\mu(H, a_H^*) > \mu(L, a)$ for all $a \in A$, while (ii) guarantees that $\mu(L, a_L^*) < \mu(H, a)$ for all $a \in A$. It follows that $\mu(L, a_L^*) \neq \mu(H, a_H^*)$ so that Condition 1 and Condition 3 are satisfied.

Proposition 5. *Assume Assumption 1 holds. In any Markov perfect equilibrium of the limit pricing game,*

[14] Asymmetric payoffs in the duopoly continuation game can easily be accommodated, to capture either asymmetric cost structure or a different equilibrium selection.

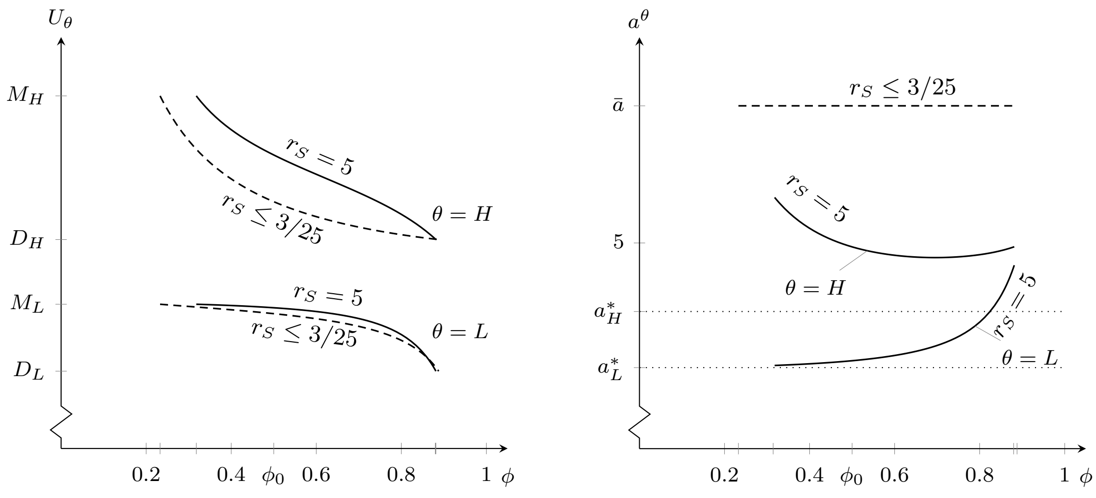

Figure 1: Left: Incumbent's value functions. Right: Incumbent's equilibrium actions. $(H, L, b, c, F, o, \sigma, r_{DM}, \bar{a}) = (100, 70, 1/20, 10, 100, 20, 4, 3/2, 7)$.

(i) both types of incumbent engage in limit pricing, that is, produce a quantity higher than the myopically optimal quantity;

(ii) in any equilibrium, the expected price is always higher when the demand is strong, i.e., $H - a_H(\phi) \geq L - a_L(\phi)$, for any $\phi \in [\underline{\phi}, \overline{\phi}]$.

Figure 1 illustrates the equilibrium value functions and the quantities for different discount rates. First, as proved in Proposition 1, the value functions are decreasing. Second, the incumbent has incentives to overproduce to induce a lower price and deter entry. As explained in Section 4, when best replying, the incumbent takes the potential entrant's conjecture and, hence, the speed of learning, as given: the higher the speed of learning and the steeper the value function, the stronger the incentives to overproduce.

As stated in (ii), in equilibrium, the expected price is always higher when the demand is strong. To put it differently, as illustrated in Figure 2, in equilibrium, γ is always positive and both types have an incentive to overproduce to put pressure on the price.

The expected price path can never cross in equilibrium: even if the firm facing a strong demand wants to overproduce so to induce a price as low as the firm facing a weak demand, by assumption the weak firm has the ability to push the price to levels unattainable to the strong firm. In the limit as the two firms become arbitrarily patient, this logic "unravels" and both firms produce at capacity.

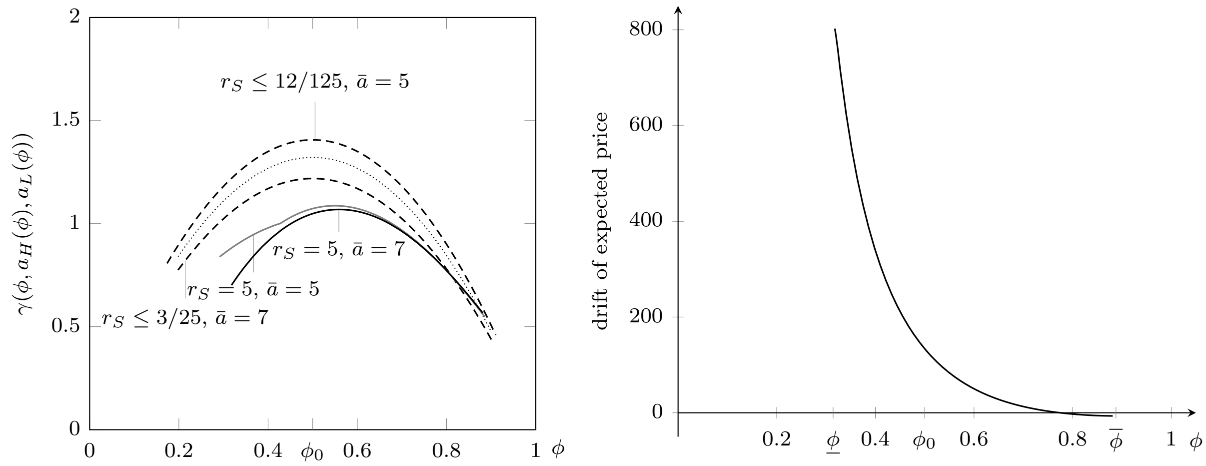

Figure 2: Left: Speed of Learning. Right: Drift of the Expected Price. $(H, L, b, c, F, o, \sigma, r_{DM}, \bar{a}) = (100, 70, 1/20, 10, 100, 20, 4, 3/2, 7)$.

In the equilibrium in Figure 1, both types ramp up output as the threat of entry grows closer. The example illustrates how our model can easily replicate the pricing patterns which have been observed in the airline industry as part of the phenomenon commonly known as the "Southwest Effect." In the 1990s and 2000s, incumbent airlines responded to the threat of entry by Southwest by lowering their prices and, as documented for example by Goolsbee and Syverson (2008), the magnitude of price cuts tended to increase over time in some markets. We shall further investigate the statistical properties of the equilibrium price process in the section.

5.2 Price Paths

Our model can inform the design of tests to detect anti-competitive behavior, in line with Bolton et al.'s (2000) advocacy for a strategic approach of antitrust authorities and courts to predatory pricing. As a first step in this direction, the following proposition identifies a statistical property that the time paths of prices must satisfy when the incumbent does not engage in an entry deterrence behavior.

Lemma 1. *If both types of incumbent produce the myopically optimal quantity, from the perspective of the potential entrant, the price is a martingale.*

In the Markov equilibrium of our game, the price is martingale at some $\phi \in (0,1)$ only if

$$2La_L'(\phi) - L(1-\phi)a_L''(\phi) = 2Ha_H'(\phi) + H\phi a_H''(\phi), \tag{5}$$

which holds when both types produce at capacity, or more in general when the equilibrium features a constant quantity path. In any Markov equilibrium featuring interior quantity this condition is unlikely to hold, as corroborated by our numerical analysis. Figure 2 illustrates a numerical example in which the equilibrium expected price is a submartingale for low beliefs and a supermartingale for high beliefs. This pattern is reminiscent of the "Southwest effect" as documented by Goolsbee and Syverson (2008): "as time passes without Southwest entering, prices fall further."

5.3 Welfare

Conceptually, one can decompose the welfare effect of limit pricing into three components. First, the welfare gains or losses before the potential entrant take the irreversible action; with regard to consumer surplus, aggressive limit pricing is always beneficial. Second, the welfare gains or losses once the potential entrant takes the irreversible action; with regard to consumer surplus, the lower the probability of entry, the larger the loss. Third, conditional on the potential entrant taking the decision that maximizes welfare, the delay entails a cost.

The welfare analysis of limit pricing has been vastly influenced by Milgrom and Roberts (1982) and has mostly focused on the gains from price cuts. For example, the estimates of the welfare effects of limit pricing in Sweeting et al. (2020) rely on the fact that, if one focuses on the separating equilibrium of Milgrom and Roberts (1982)—as commonly done in empirical applications—entry decisions would be the same under either complete or asymmetric information.

While in Matthews and Mirman (1983) and in the pooling equilibrium of Milgrom and Roberts (1982) the welfare effects can be negative because entry may be successfully threatened, unless it reduces the probability of entry, limit pricing is always beneficial. This section is aimed at highlighting a novel welfare tradeoff that emerges once the intensive time-margin delay is taken into account.

Compared to Milgrom and Roberts (1982) and Matthews and Mirman (1983)'s two-period models, in our model, the potential entrant faces a trade-off between a more informed decision (that is, waiting and gathering more data about the market conditions before acting) and discounting (that is, foregoing profits by not acting early on). Further, the incumbent pricing strategy affects the precision of the information that the potential entrant has access to before taking its decision.

We divide our analysis of the welfare tradeoffs in two sections. First, we presents a few analytical results regarding the probability of entry and the delay. Second, we illustrate numerically when limit pricing can hurt consumer.

5.3.1 Deterrence Probability and Entry Delay

As mentioned in the introduction, one can interpret $\bar{a}$ as an output restriction rule, in the spirit of the rule proposed by Williamson (1977) and Edlin (2002) and recently analyzed by Rey et al. (2024), to mitigate predatory behavior. While restricting output reduces the consumer surplus by curtailing the price cuts, the effects on the probability of entry and the delay are a priori unclear.

In fact, Figure 2 shows that a potential entrant facing a patient incumbent has access to a (uniformly) less or more informative signal, as compared to the case when the incumbent does not engage in limit pricing, depending on the magnitude of $\bar{a}$. The lemma below identifies conditions for either case to apply.

Lemma 2. *The price is more informative when both types produce $\bar{a}$, as compared to when they produce the monopoly quantity, if and only if $\bar{a} < (Ha_H^* - La_L^*)/(H - L)$.*

Traditional theories of *test market predation*, as well as the most cited cases,[15] concern situations in which the incumbent's limit pricing strategy reduces the information acquired by the potential entrant. In principle, however, aggressive pricing may allow the entrant to learn about the relevant segments of the market, or the demand at the relevant prices.[16] While parsimonious, our model captures both the scenario in which aggressive pricing decreases the informativeness of the signal and the scenario in which the opposite is true.

To understand how the informativeness of the price ultimately affects welfare, we first consider the case in which the potential entrant only decides when to enter, or

[15] "In the 1980s P&G tried to get into the bleach business. [...] We went to test-market in Portland, Maine. [...] Do you know what Clorox did? They gave every household in Portland, Maine, a free gallon of Clorox bleach—delivered to the front door. "

[16] For example, in the airline industry, the low-end market had been generally ignored by major carriers up until the entry or threat of entry of low-cost carriers (Brady and Cunningham, 2001).

equivalently, $o = 0$,[17] and focus on the case of a patient incumbent. In this case, in the unique equilibrium, both types of incumbent produce at capacity and the potential entrant always enters when the demand is strong, so at least as far as the second and third components of the welfare effect are concerned, we can focus of the study of two statics: the unconditional probability of entry, and the expected discount factor, a measure of the (inverse of the) cost of delay.

Proposition 6. *Suppose the regulator can choose $\bar{a} \in \overline{A}$, for some closed $\overline{A} \subset (a^*_H, (H-L)/(bH) + H/L \cdot L/(c+2bL))$ and the incumbent is patient enough.*

(i) The output restriction rule that maximizes the probability of entry is $\max \overline{A}$.

(ii) Assume $\phi_0 = 1/2$, $D_H + D_L < 0$. The output restriction rule that maximizes the potential entrant's expected discount factor $\mathbf{E}[e^{-r_E \tau}]$ (minimizes the cost of delay) is $\min \overline{A}$.

The proposition sheds light on the tradeoffs faced by a regulator designing an output restriction rule. On the one hand, when delay is likely to harm consumers, either because of impatience or via (unmodeled) lower product quality and more limited choice offered in monopoly, a conservative output restriction rule may be beneficial. On the other hand, a patient regulator may tolerate aggressive limit pricing to reap the benefit of deep price cuts. Hence, depending on the exact objective of the regular, output restriction rules may or may not be beneficial.

5.3.2 Numerical Simulations

Figure 3 plots the consumer welfare, expected discount factor, and the ex ante probability of entry for different parameters of the model. In the figure, the consumer welfare is decomposed into three components: in blue and red, the expected discounted consumer welfare after the potential entrant takes its irreversible action, conditional on the demand being strong or weak, respectively; in yellow, the expected discounted consumer welfare before the potential entrant acts. On all panels, we plot on the left the case when the incumbent is sufficiently patient, so that the unique equilibrium involves production at capacity for both types; in the center, the case of an intermediate discount rate; on the right, the case when the incumbent does not engage in limit pricing.

[17] Our equilibrium characterization generalizes to the case of one-sided action with minor adjustments, as explained more in detail in Section 6.

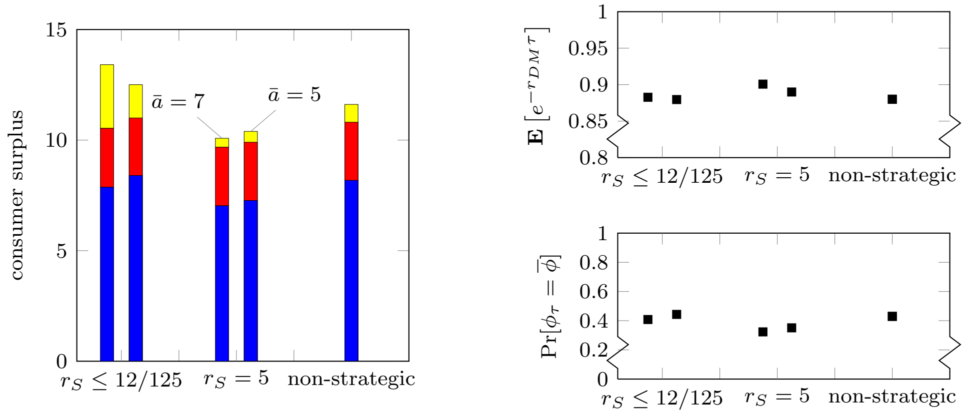

Figure 3: Consumer surplus discounted at rate 3/2, expected discount factor, and probability of entry . In yellow, the expected discounted consumer surplus between $[0, \tau)$. In blue (red) the expected discounted (lump-sum) consumer surplus at τ when $\theta = H$ ($\theta = L$). $(H, L, b, c, F, o, \sigma, r_{DM}) = (100, 70, 1/20, 10, 100, 20, 4, 3/2)$.

When the discount rate of the incumbent is intermediate, the equilibrium involves interior actions, and limit pricing hurts consumer surplus. Perhaps surprisingly the welfare loss is larger when the capacity constraint is lower. When the capacity constraint is lower, in equilibrium, the price is more informative and the potential entrant is more likely to enter.

The effect of the capacity constraint is reversed if we look at the case where the incumbent is sufficiently patient. In both cases, in the unique equilibrium, both types of incumbent produce the maximum feasible quantity and the consumers gain from limit pricing. Because the gains are mostly coming from lower pricing, the higher the maximum feasible quantity, the larger the gains in consumer surplus.

The conclusion we draw in this section resonates with the empirical evidence. For example, in the 1970s, Folger (owned by Procter & Gamble) delayed by several years entry in the Eastern United States regular coffee market because General Foods reacted to this threat by sharply reducing the price of its Maxwell House.[18,19] Procter & Gamble had a practice of conducting market tests before undertaking large-scale entry but Maxwell's aggressive pricing behavior affected the informativeness of early

[18] See Hilke and Nelson (1989) and Ross and Scherer (1990).

[19] Traditional industrial organization distinguishes between limit pricing, that is, pricing strategy to discourage entry, and predatory pricing, that is pricing strategy to encourage exit. However, our limit pricing model resonates with the theories of test market predation, according to which an incumbent pricing strategy affects the information acquired by a competitor and deters expansion.

tests. While the FTC dismissed the complaint,[20] the case is frequently cited as an example of test market predation (see, Viscusi et al., 2018, Ch. 8).

In line with the insights provided by existing models, the FTC based its decision in that case on the finding that the alleged predator did not have a dangerous probability of success (see, Hilke and Nelson, 1989). We believe that our model provides a new theoretical perspective for reexamining the court's approaches toward predatory pricing and attempted monopolization.

6 Promotion

Our second application considers a promotion model in the spirit of Fairburn and Malcomson (2001) in which a firm decides whether to promote a worker based on the observed performance. This application also illustrates how one can relax some of the assumptions maintained in the main body of the paper.

The sender is an agent who has private information about his ability, $\theta \in \{H, L\}$. The DM is an employer who decides whether and when to promote the agent, and would like to promote the agent only if $\theta = L$. This is the first point of departure from the main model: the DM can only take one action, that is, he chooses only τ, i.e., by assumption $b_\tau = l$ whenever $\tau < \infty$, and $G(H, l) > 0 > G(L, l)$.

At each point in time, the agent chooses an effort level $a_t \in [\underline{a}, \bar{a}] \subset \mathbf{R}_+$, and the employer observes a noisy signal of the agent's ability and effort,

$$\mathrm{d}X_t = (\bar{\mu} - \theta a_t)\,\mathrm{d}t + \sigma\,\mathrm{d}Z_t,$$

where $L > H$. The noisy signal can be interpreted as the number of customer complaints: the more effort the agent puts in, the lower the number of complaints. The agent of type L has a higher ability in that the marginal return of his effort in reducing the number of complaints is higher, which justifies the employer's preferences.

The agent faces a cost of effort, which for simplicity we assume to be quadratic, and is rewarded according to a pay-for-performance compensation at rate β. As a result, his expected flow payoff $\pi(\theta, a) = \beta\theta a - ca^2/2$, where $c > 0$.

The second point of departure from the model in the main body of the paper concerns the payoff the agents collect if promoted. We assume that $\Pi(\theta, l) > \pi^*(\theta)$.

[20] *General Foods Corporation*, 103 F.T.C. 204.

To guarantee that the baseline assumptions, as well as Condition 1, Condition 2, and Condition 3 are satisfied in this model, we impose the following parametric restrictions.

Assumption 2. *The set of feasible actions is* $A = [\underline{a}, \bar{a}]$, $\underline{a} > \bar{a}$, $\underline{a} \in (\beta H^2/(cL), \beta H/c)$ *and* $\bar{a} \in (\beta L/c, \beta L^2/(cH))$.

In the Appendix, we show that Assumption 2 implies the desired conditions. Our equilibrium characterization generalizes to the case of one-sided action with minor adjustments. The following proposition collects the counterparts of the equilibrium properties derived in Section 4.

Proposition 7. *In any Markov perfect equilibrium of the promotion game,*

(i) the value functions of both types of agent are weakly decreasing, i.e., $U'_\theta(\phi) \leq 0$ *for all* $\phi \in (\underline{\phi}, 1)$;

(ii) both types of agent choose a level of effort higher than the myopically optimal level, i.e., $a_\theta(\phi) \geq a^*_\theta$ *for all* $\phi \in (\underline{\phi}, 1)$.

(iii) the agent always gets promoted when the $\theta = L$ *and may get promoted if* $\theta = H$.

In this simple model, the employer is incentivizing effort using both pay-per-performance and promotion. As a result, an inefficient agent is promoted with positive probability, which is reminiscent of the Peter Principle.[21]

The next proposition formalizes the idea that a "naive" agent can achieve a higher payoff than a "strategic" one.

Proposition 8. *Assume* $\bar{a} < \beta(H + L)/c$. *The average expected ex ante payoff of the agent in any Markov perfect equilibrium of the promotion game is lower compared to the case when the set of feasible actions is restricted to the myopically optimal action; that is,* $A_\theta = \{a^*_\theta\}$, *provided that the agent is patient enough.*

In equilibrium, a sufficiently patient agent exerts maximal level of effort. Expecting the signal to be more informative, the employer raises the standard he adopts to

[21] As explained by Fairburn and Malcomson (2001), "if a firm provides incentives by promoting those who have performed well in a job, it may simply transfer them to a job to which they are not well suited, a mild version of the Peter Principle." The Peter Principle, originally introduced by Peter and Hull (1969), asserts that organizations promote people who are good at their jobs until these employees reach their "level of incompetence".

grant promotion to the agent, i.e., uses a lower cutoff belief $\underline{\phi}$. While the agent of type L always gets promoted, a lower promotion belief $\underline{\phi}$ reduces the probability that an agent of type H is promoted. This does not necessarily imply that an agent would be better off if he could, before knowing his type, commit to exerting the myopically optimal level of effort, because the expected time before the employer promotes the agent may be longer in this case. However, if the agent is patient enough, the cost of delay is negligible and hence naivete is beneficial.

7 Conclusion

While our analysis is mainly theoretical, our two applications illustrate how our tractable characterization can be leveraged to conduct comparative static exercises and inform policy interventions.

In the paper, we focused on a canonical setup, but the framework can be easily extended in a few directions. Generalizing the result to allow for multidimensional Gaussian signals or for additional conclusive Poisson signals is straightforward.

We assumed that the DM collects payoffs only upon taking the irreversible action. The extension to the case in which the DM also collects payoffs before does not add conceptual difficulties but is not immediate especially when the payoffs depend on the the state and the action of the sender. In the context of dynamic competition, this extension would allow us to talk about predation, that is, about a dominant firm trying to induce the exit of a competitor.

A more sophisticated predation model, however, would allow for investment in capacity expansion. For example, airlines can invest in their fleet to increase their capacity, or workers can invest in education to improve their skills, leading to endogenously evolving types. This could be captured, for example, by allowing the sender to affect the evolution of the state, as in Board and Meyer-ter-Vehn (2013). We are pursuing this in ongoing work.

References

Amster, Pablo (2007) "Nagumo and Landesman-Lazer type conditions for nonlinear second order systems," *Nonlinear Differential Equations and Applications NoDEA*, 13, 699–711.

Anderson, Axel and Lones Smith (2013) "Dynamic Deception," *American Economic Review*, 103, 2811–47.

Atakan, Alp E. and Mehmet Ekmekci (2012) "Reputation in Long-Run Relationships," *The Review of Economic Studies*, 79, 451–480.

Atakan, Alp E. and Mehmet Ekmekci (2015) "Reputation in The Long-Run with Imperfect Monitoring," *Journal of Economic Theory*, 157, 553–605.

Board, Simon and Moritz Meyer-ter-Vehn (2013) "Reputation for Quality," *Econometrica*, 81, 2381–2462.

Bohren, J. Aislinn (2024) "Persistence in a Dynamic Moral Hazard Game," *Theoretical Economics*, 19, 449–498.

Bolton, Patrick, Joseph F. Brodley, and Michael H. Riordan (2000) "Predatory Pricing: Strategic Theory and Legal Policy," *Georgetown Law Journal*, 88, 2239–2330.

Bonatti, Alessandro, Gonzalo Cisternas, and Juuso Toikka (2017) "Dynamic Oligopoly with Incomplete Information," *The Review of Economic Studies*, 84, 503–546.

Borodin, Andrei N. and Paavo Salminen (2015) *Handbook of Brownian Motion: Facts and Formulae*, Basel: Birkhäuser, 2nd edition.

Brady, Stephan P. and William A. Cunningham (2001) "Exploring Predatory Pricing in the Airline Industry," *Transportation Journal*, 41, 5–15.

Brocas, Isabelle and Juan D. Carrillo (2007) "Influence through Ignorance," *The RAND Journal of Economics*, 38, 931–947.

Celetani, Marco, Drew Fudenberg, David K. Levine, and Wolfgang Pesendorfer (1996) "Maintaining a Reputation Against a Long-Lived Opponent," *Econometrica*, 64, 691–704.

Cetemen, Doruk (2021) "Efficiency in Repeated Partnerships," Working Paper.

Chernoff, Herman (1972) "Sequential Analysis and Optimal Design," in *CBMS-NSF Regional Conference Series in Applied Mathematics*, Vol. 8, 1–120, Philadelphia: Society for Industrial and Applied Mathematics.

Cisternas, Gonzalo (2018) "Two-sided learning and the ratchet principle," *The Review of Economic Studies*, 85, 307–351.

Cisternas, Gonzalo and Aaron Kolb (2024) “Signaling with Private Monitoring,” *Review of Economic Studies*, forthcoming.

Cripps, Martin W., George J. Mailath, and Larry Samuelson (2004) “Imperfect Monitoring and Impermanent Reputations,” *Econometrica*, 72, 407–432.

Cripps, Martin W. and Jonathan P. Thomas (1997) “Reputation and Perfection in Repeated Common Interest Games,” *Games and Economic Behavior*, 18, 141–158.

Daley, Brendan and Brett Green (2012) “Waiting for News in the Market for Lemons,” *Econometrica*, 80, 1433–1504.

Dilmé, Francesc (2019) “Dynamic Quality Signaling with Hidden Actions,” *Games and Economic Behavior*, 113, 116–136.

Dilmé, Francesc (2024) “Repeated Bargaining with Imperfect Information about Previous Transactions,” Working Paper.

Dixit, Robert K. and Robert S. Pindyck (1994) *Investment under Uncertainty*, Princeton: Princeton University Press.

Edlin, Aaron S. (2002) “Stopping Above-Cost Predatory Pricing,” *The Yale Law Journal*, 111, 941–991.

Ekmekci, Mehmet, Leandro Gorno, Lucas Maestri, Jian Sun, and Dong Wei (2022) “Learning from Manipulable Signals,” *American Economic Review*, 112, 3995–4040.

Ekmekci, Mehmet and Lucas Maestri (2022) “Wait or Act Now? Learning Dynamics in Stopping Games,” *Journal of Economic Theory*, 205, 105541.

Faingold, Eduardo and Yuliy Sannikov (2011) “Reputation in Continuous-Time Games,” *Econometrica*, 79, 773–876.

Fairburn, James A. and James M. Malcomson (2001) “Performance, Promotion, and the Peter Principle,” *The Review of Economic Studies*, 68, 45–66.

Goolsbee, Austan and Chad Syverson (2008) “How Do Incumbents Respond to the Threat of Entry? Evidence from the Major Airlines,” *The Quarterly Journal of Economics*, 123, 1611–1633.

Gryglewicz, Sebastian and Aaron Kolb (2022) “Dynamic Signaling with Stochastic Stakes,” *Theoretical Economics*, 17, 539–559.

Gryglewicz, Sebastian and Aaron Kolb (2023) "Strategic Pricing in Volatile Markets," *Operations Research*, forthcoming.

Henry, Emeric and Marco Ottaviani (2019) "Research and The Approval Process: The Organization Of Persuasion," *American Economic Review*, 109, 911–55.

Hilke, John C. and Philip B. Nelson (1989) "Strategic Behavior and Attempted Monopolization: The Coffee (General Foods) Case," in *The Antitrust Revolution* ed. by Jr. John E. Kwoka and Lawrence J. White. Glenview, Ill.: Scott, Foresman.

Karatzas, Ioannis and Steven Shreve (1996) *Brownian Motion and Stochastic Calculus*, New York: Springer.

Keller, Godfrey and Sven Rady (1999) "Optimal Experimentation in a Changing Environment," *The Review of Economic Studies*, 66, 475–507.

Kolb, Aaron M. (2015) "Optimal Entry Timing," *Journal of Economic Theory*, 157, 973–1000.

Kolb, Aaron M. (2019) "Strategic Real Options," *Journal of Economic Theory*, 183, 344–383.

Kuvalekar, Aditya and Elliot Lipnowski (2020) "Job insecurity," *American Economic Journal: Microeconomics*, 12, 188–229.

Lamberton, Damien and Mihail Zervos (2013) "On the Optimal Stopping of a One-Dimensional Diffusion," *Electronic Journal of Probability*, 18, 1–49.

Liptser, Robert S. and Albert N. Shiryaev (2001) *Statistics of Random Processes: I. General Theory*, Berlin, Heidelberg: Springer.

Matthews, Steven A. and Leonard J. Mirman (1983) "Equilibrium Limit Pricing: The Effects of Private Information and Stochastic Demand," *Econometrica*, 51, 981–996.

McClellan, Andrew (2022) "Experimentation and Approval Mechanisms," *Econometrica*, 90, 2215–2247.

Milgrom, Paul and John Roberts (1982) "Limit Pricing and Entry under Incomplete Information: An Equilibrium Analysis," *Econometrica*, 50, 443–459.

Moscarini, Giuseppe and Lones Smith (2003) "The Optimal Level of Experimentation," *Econometrica*, 69, 1629–1644.

Ok, Efe A. (2007) *Real Analysis with Economic Applications*, Princeton: Princeton University Press.

Orlov, Dmitry, Andrzej Skrzypacz, and Pavel Zryumov (2020) "Persuading the Principal to Wait," *Journal of Political Economy*, 128, 2542–2578.

Pei, Harry (2021) "Trust and Betrayals: Reputational Payoffs and Behaviors Without Commitment," *Theoretical Economics*, 16, 449–475.

Peter, Laurence J. and Raymond Hull (1969) *The Peter Principle: Why Things Always Go Wrong*, New York: William Morrow & Co.

Rey, Patrick, Yossi Spiegel, and Konrad Stahl (2024) "A Dynamic Model of Predation," Working Paper.

Ross, David and Frederic M. Scherer (1990) *Industrial Market Structure and Economic Performance*, Boston: Houghton Mifflin, 3rd edition.

Salas, Christian (2019) "Persuading policy-makers," *Journal of Theoretical Politics*, 31, 507–542.

Saloner, Garth (1984) "Dynamic Equilibrium Limit-Pricing in an Uncertain Environment," Working Paper.

Stokey, Nancy L. (2009) *The Economics of Inaction: Stochastic Control Models with Fixed Costs*, Princeton: Princeton University Press.

Sweeting, Andrew, James W. Roberts, and Chris Gedge (2020) "A Model of Dynamic Limit Pricing with an Application to the Airline Industry," *Journal of Political Economy*, 128, 1148–1193.

Tirole, Jean (1988) *The Theory of Industrial Organization*, Cambridge, MA: MIT Press.

Toxvaerd, Flavio (2017) "Dynamic Limit Pricing," *The RAND Journal of Economics*, 48, 281–306.

Viscusi, W. Kip, Joseph E. Harrington, and David E. M. Sappington (2018) *Economics of Regulation and Antitrust*, Cambridge, MA: MIT Press, 5th edition.

Wald, Abraham (1945) "Sequential Tests of Statistical Hypotheses," *The Annals of Mathematical Statistics*, 16, 117–186.

Williamson, Oliver E. (1977) "Predatory Pricing: A Strategic and Welfare Analysis," *The Yale Law Journal*, 87, 284–340.

8 Appendix

8.1 Bounds on Coefficient γ

This section proves a bound on γ which is used in the subsequent analysis.

Lemma 3. *Assume Condition 1 and Condition 2. There exists a $C > 0$ such that for all $(a_H, a_L, \phi, z_H, z_L) \in A \times A \times (0,1) \times \mathbf{R} \times \mathbf{R}$, if $(a_H, a_L,) \in \mathcal{N}(\phi, z_H, z_L)$, then*

$$(1 + |z_H| + |z_L|) \frac{|\gamma(a_H, a_L, \phi)|}{\phi(1-\phi)} \geq C.$$

Proof. Suppose that such a constant does not exist. Then, there exists a sequence $(a_{H,n}, a_{L,n}, \phi_n, z_{L,n}, z_{H,n})_{n\in\mathbf{N}}$ with $\phi_n \in (0,1)$ and $(a_{H,n}, a_{L,n}) \in \mathcal{N}(\phi_n, z_{H,n}, z_{L,n})$, such that for both $\theta = H$ and $\theta = L$ the following hold:

$$|z_{\theta,n}| \frac{|\gamma(a_{H,n}, a_{L,n}, \phi_n)|}{\phi_n(1-\phi_n)} \to 0, \quad \text{and} \quad \frac{|\gamma(a_{H,n}, a_{L,n}, \phi_n)|}{\phi_n(1-\phi_n)} \to 0. \tag{6}$$

By compactness, there exists a sub-sequence converging to some $(a_H, a_L, \phi, z_L, z_H) \in A \times A \times [0,1] \times \mathbf{R} \times \mathbf{R}$. By continuity, (a_H, a_L, ϕ) must be a Bayes Nash equilibrium of the auxiliary signaling game with prior ϕ, i.e., $(a_H, a_L) \in \mathcal{N}(\phi, z_L, z_H)$.

Hence, the first limit in (6) implies that $a_{H,n} \to a_H^*$ and $a_{L,n} \to a_L^*$. Let $\varepsilon = |\mu(H, a_H^*) - \mu(L, a_L^*)|$. By Condition 2 and the continuity of μ, for any n sufficiently high, $|\mu(H, a_{H,n}) - \mu(L, a_{L,n})| \geq \varepsilon/2 > 0$, contradicting the second limit in (6). □

The following corollary implies that our system satisfies a quadratic growth condition for each type of the sender.

Corollary 2. *Assume Condition 1 and Condition 2. For all $\varepsilon > 0$, there exist a $K > 0$ such that for all $\phi \in [\varepsilon, 1-\varepsilon]$, $(U_H, U_L) \in [\Pi(H,l), \Pi(H,h)] \times [\Pi(L,l), \Pi(L,h)]$,* *and* $(U_H', U_L') \in \mathbf{R}_-$, *if* $(a_H, a_L) \in \mathcal{N}(\phi, \phi(1-\phi)U_H'(\phi)/r_S, \phi(1-\phi)U_L'(\phi)/r_S)$ *then*

$$\left| -2\frac{U_H'(\phi)}{\phi} + \frac{2r_S\left(U_H(\phi) - \pi(H, a_H(\phi))\right)}{\left(\gamma\left(a_H(\phi), a_L(\phi), \phi\right)\right)^2}\right| \le K\left(1 + \left(U_H'(\phi)\right)^2 + \left(U_L'(\phi)\right)^2\right),$$
$$\left|2\frac{U_L'(\phi)}{1-\phi} + \frac{2r_S\left(U_L(\phi) - \pi(L, a_L(\phi))\right)}{\left(\gamma\left(a_H(\phi), a_L(\phi), \phi\right)\right)^2}\right| \le K\left(1 + \left(U_H'(\phi)\right)^2 + \left(U_L'(\phi)\right)^2\right).$$

The proof follows directly from the bounds derived in Lemma 3.

8.2 Proof for Section 3

8.2.1 Proof of Proposition 1

The proof relies on a modification of Theorem 2.1 of Amster (2007), which we state and prove in the Online Appendix. To apply Theorem OA.1, define the constant functions $\alpha : [0,1] \to \mathbf{R}^2$, $\alpha \equiv (\underline{\pi}, \underline{\pi})$, and $\beta : [0,1] \to \mathbf{R}^2$, $\beta \equiv (\pi^*(H), \pi^*(L))$.

Let $f : [0,1] \times \mathbf{R}^4 \to \mathbf{R}^2$ be defined as

$$f_1(\phi, U, U') = -2\frac{U_1'}{\phi} + 2r_S\frac{U_1 - \pi\left(H, \text{proj}_1\mathcal{N}\left(\phi, \phi(1-\phi)U_1'/r_S, \phi(1-\phi)U_2'/r_S\right)\right)}{\left(\gamma\left(\mathcal{N}\left(\phi, \phi(1-\phi)U_1'/r_S, \phi(1-\phi)U_2'/r_S\right), \phi\right)\right)^2},$$
$$f_2(\phi, U, U') = 2\frac{U_2'}{1-\phi} + 2r_S\frac{U_2 - \pi\left(L, \text{proj}_2\mathcal{N}\left(\phi, \phi(1-\phi)U_1'/r_S, \phi(1-\phi)U_2'/r_S\right)\right)}{\left(\gamma\left(\mathcal{N}\left(\phi, \phi(1-\phi)U_1'/r_S, \phi(1-\phi)U_2'/r_S\right), \phi\right)\right)^2}.$$

Consider the following boundary value problem

$$U''(\phi) = f(\phi, U(\phi), U'(\phi)), \qquad \phi \in [\underline{\phi}, \bar{\phi}],$$
$$U(\underline{\phi}) = (\Pi(H, l), \Pi(L, l)), \qquad U(\bar{\phi}) = (\Pi(H, h), \Pi(L, h)).$$

It can be verified that α and β are a lower and an upper solution of the boundary value problem, respectively. By Corollary 2, there exist a constant $K > 0$ such that

$$|f_i(\phi, U_i(\phi), U_i'(\phi))| \le K\left(1 + \left(\left|U_1'(\phi)\right| + \left|U_2'(\phi)\right|\right)^2\right)$$

for $i = 1, 2$ and for all $(\phi, U, U') \in [\underline{\phi}, \bar{\phi}] \times [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \mathbf{R}_-^2$. For each $i = 1, 2$, let $\psi_i(s) = K(1 + s^2)$. Hence, one can choose M_1 and M_2 such that

$$\int_{r_i}^{M_i} \frac{1}{K}\frac{s}{1+s^2}\,\mathrm{d}s > 2\max\{\alpha_H, \alpha_L\},$$

so to satisfy the Nagumo-type condition in Theorem OA.1, where $r_1 := (\Pi(H,l) - \Pi(H,h))/(\overline{\phi} - \underline{\phi})$ and $r_2 := (\Pi(L,l) - \Pi(L,h))/(\overline{\phi} - \underline{\phi})$. Further, it is readily verified that condition (OA.3) in Theorem OA.1 holds with $U_i = \underline{U}$ for $i = 1, 2$, see (4). We can conclude that the boundary value problem has a bounded $\mathcal{C}^2$ solution in the domain $\left[\underline{\phi}, \overline{\phi}\right]$ taking values in $[\underline{\pi}, \Pi(H,l)) \times [\underline{\pi}, \Pi(L,l))$.

8.2.2 Proof of Proposition 2

Given a regular strategy profile for the sender, the DM faces a standard optimal stopping problem. He maximizes $\mathbf{E}\left[e^{-r_{DM}\tau} g(\phi_\tau) \mathbf{1}_{\tau<\infty}\right]$ over all stopping times τ, where $g(\phi) := \arg\max\{\phi_\tau G(H,h) + (1-\phi_\tau)\, G(L,h), \phi_\tau G(H,l) + (1-\phi_\tau)\, G(L,l)\}$. By Theorem 7.5 in Lamberton and Zervos (2013), the optimal value function admits the following characterization

$$V(\phi) = \inf\{A\varphi + B\psi \mid A, B \geq 0 \text{ and } A\varphi(\phi) + B\psi(\phi) \geq g(\phi) \text{ for all } \phi \in (0,1)\},$$

where φ and ψ are the fundamental solutions to the ODE

$$r_{DM} V(\phi) = \frac{1}{2} \left(\gamma(a_H, a_L, \phi)\right)^2 V_{\phi\phi}(\phi), \tag{7}$$

that is, the functions ϕ and ψ are $\mathcal{C}^1$, their first derivatives are absolutely continuous functions, $0 < \phi(x)$ and $\phi'(x) < 0$ for all $x \in (0,1)$, $0 < \psi(x)$ and $\psi'(x) > 0$ for all $x \in (0,1)$, $\lim_{x\downarrow 0} \psi(x) = \lim_{x\uparrow 1} \phi(x) = 0$, and $\lim_{x\downarrow 0} \phi(x) = \lim_{x\uparrow 1} \psi(x) = \infty$. (See Borodin and Salminen, 2015, II.1.)

Furthermore, if $\underline{\phi} < \bar{\phi}$ are any points in $(0,1)$ such that $V(\phi) > g(\phi)$ for all $\phi \in (\underline{\phi}, \bar{\phi})$, then there exist constants $\tilde{A}, \tilde{B}$ such that $V(\phi) = \tilde{A}\varphi(\phi) + \tilde{B}\psi(\phi)$ for all $\phi \in (\underline{\phi}, \bar{\phi})$ and $\tilde{A}\varphi(\phi) + \tilde{B}\psi(\phi) \geq g(\phi)$ for all $\phi \in (0,1)$.

Also, by Corollary 7.5 in Lamberton and Zervos (2013), the value function satisfies smooth pasting, so it solves the boundary value problem in the statement of the proposition.

8.2.3 Proof of Theorem 1

Lemma OA.3, relegated to the online appendix, proves the continuity of the solutions to the boundary value problem of the sender with respect to the boundary conditions.

For any ordered pair $(\underline{\phi}, \bar{\phi}) \in [0,1]^2$, define the sender's best-reply mapping $\text{BR}^S : (\underline{\phi}, \bar{\phi}) \mapsto (a_H(\phi), a_L(\phi)) \in (A \times A)^{[0,1]}$ by pasting together a solution to the system of ordinary differential equations in Proposition 1, which specifies the value of the function for $\phi \in [\underline{\phi}, \bar{\phi}]$, and the constant functions (a_H^*, a_L^*). By Lemma OA.3, we can assume that BR^S is continuous.

Similarly, for any regular strategy profile $(a_H, a_L) : [0,1] \to A^2$ define the DM's best-reply mapping $\text{BR}^{DM} : (A \times A)^{[0,1]} \mapsto [0,1]^2$ as the unique pair of cutoffs characterizing the DM's best reply.

The best reply of the DM is continuous: by the characterization in the proof of 2, and since the fundamental solutions to the ODE (7) are continuous in the action profile of the sender, both the value function and the optimal cutoffs of the DM are continuous in the action profile of the sender.

Define the mapping $\Gamma : [0,1]^2 \to [0,1]^2$ by combining the two best replies, that is, $\Gamma : (\underline{\phi}, \bar{\phi}) \mapsto \text{BR}^{DM}(\text{BR}^S(\underline{\phi}, \bar{\phi}))$. Since the composition of the continuous functions, Γ is continuous. Therefore, by Brouwer's fixed-point theorem, Γ has a fixed point. By construction, any fixed point is a Markov Perfect equilibrium.

8.3 Proof for Section 4

8.3.1 Proof of Proposition 3

Follows directly from the proof of Proposition 1, as Theorem OA.1 proves the existence of a monotone bounded solution to the boundary value problem.

8.3.2 Proof of Proposition 4

Proof of (i) By Lemma 3, in any equilibrium, $\gamma(a_H(\phi), a_L(\phi), \phi)$ is bounded away from zero; that is, in any equilibrium $\gamma(a_H(\phi), a_L(\phi), \phi)$ cannot change sign in the interval $[\underline{\phi}, \bar{\phi}]$.

Reasoning by contradiction, assume that $\gamma(a_H(\phi), a_L(\phi), \phi) < 0$, that is $\mu(H, a_H(\phi)) - \mu(L, a_L(\phi)) < 0$, for all $\phi \in (\underline{\phi}, \bar{\phi})$. Because, by Proposition 3, $U_\theta'(\phi) \leq 0$, the local incentive constraint (3) implies that $a_H(\phi) \le a_H^*$ and $a_L(\phi) \le a_L^*$. For any $\phi \in [\underline{\phi}, \bar{\phi}]$

$$
\begin{aligned}
0 > \mu(H, a_H(\phi)) - \mu(L, a_L(\phi)) &\ge -\int_{a_H(\phi)}^{a_H^*} \mu_a(H, a)\,\mathrm{d}a + \int_{a_L(\phi)}^{a_L^*} \mu_a(L, a)\,\mathrm{d}a \\
&\ge -\int_{a_H(\phi)}^{a_H^*-a_L^*+a_L(\phi)} \mu_a(H, a)\,\mathrm{d}a - \int_{a_H^*-a_L^*+a_L(\phi)}^{a_H^*} \mu_a(L, a)\,\mathrm{d}a + \int_{a_L(\phi)}^{a_L^*} \mu_a(L, a)\,\mathrm{d}a \\
&\ge -\int_{a_H(\phi)}^{a_H^*-a_L^*+a_L(\phi)} \mu_a(H, a)\,\mathrm{d}a \ge -\mu_a(H, a_H(\phi))\,(a_H^* - a_L^* + a_L(\phi) - a_H(\phi))
\end{aligned}
$$

where the first inequality follows from part (d), the second from part (c), the third from the fact that, by (b), $a_H^* > a_L^*$ and, by (d), $\mu_{aa}(\theta, a) \le 0$, and the last inequality from (d). Hence, $a_L^* - a_L(\phi) \ge a_H^* - a_H(\phi)$, which implies $a_H(\phi) > a_L(\phi) \ge \underline{a}$. It follows from (b), the fact that, by (d), $\mu_{aa}(\theta, a) \le 0$, and the fact that $a_H(\phi) > a_L(\phi)$,

$$
\frac{\pi_a(H, a_H(\phi))}{\mu_a(H, a_H(\phi))} \ge \frac{\pi_a(L, a_L(\phi))}{\mu_a(L, a_L(\phi))}.
$$

Because $\underline{a} < a_H(\phi) \le a_H^* < \bar{a}$, the first-order conditions must hold for type H, so we obtain that at any $\phi \in [\underline{\phi}, \bar{\phi}]$,

$$
\begin{aligned}
-U_H'(\phi) &= \frac{\pi_a(H, a_H(\phi))}{\mu_a(H, a_H(\phi))} \frac{r_S \sigma}{\gamma\,(a_H(\phi), a_L(\phi), \phi)} \\
&\qquad \le \frac{\pi_a(L, a_L(\phi))}{\mu_a(L, a_L(\phi))} \frac{r_S \sigma}{\gamma\,(a_H(\phi), a_L(\phi), \phi)} \le -U_L'(\phi),
\end{aligned}
$$

where the last inequality follows from the optimality of type L's action.

However, this contradicts the boundary conditions which imply that

$$
\Pi(H, l) - \Pi(H, h) = -\int_{\underline{\phi}}^{\bar{\phi}} U_H'(\phi)\,\mathrm{d}\phi > -\int_{\underline{\phi}}^{\bar{\phi}} U_L'(\phi)\,\mathrm{d}\phi = \Pi(L, l) - \Pi(L, h).
$$

Hence, we conclude that $\gamma(a_H(\phi), a_L(\phi), \phi)$ must be positive.

Proof of (ii) Assume by contradiction that $\gamma\,(a_H(\phi), a_L(\phi), \phi) < 0$ for some $\phi \in (\underline{\phi}, \bar{\phi})$. By the local incentive constraints (2), since the drift is decreasing in the action of the sender, and by Proposition 3 above, $a_H(\phi) \le a_H^*$ and $a_L(\phi) \le a_L^*$. This, together with $a_L(\phi) \geq \underline{a}$, implies that $\mu(H, a_H(\phi)) - \mu(L, a_L(\phi)) \geq \mu(H, a_H^*) - \mu(L, \underline{a}) \geq 0$, which contradicts the $\gamma(a_H(\phi), a_L(\phi), \phi) < 0$.

8.3.3 Proof of Theorem 2

We start with two technical lemmas which are used later.

Lemma 4. *Consider a sequence of* $\{r_{S,n}, r_{DM,n}\}_{n\in\mathbf{N}}$ *together with an associated sequence of equilibria and corresponding value functions* $\{U_{H,n}, U_{L,n}\}_{n\in\mathbf{N}}$ *and cutoffs* $\{\underline{\phi}_n, \bar{\phi}_n\}_{n\in\mathbf{N}}$. *Set* $\mathcal{S} = \left\{\{\phi_n\}_{n=1,2,\ldots} : \phi_n \in [\underline{\phi}_n, \bar{\phi}_n]\right\}$. *That is,* $\mathcal{S}$ *is the set of sequences of beliefs such that for each* n*, the belief* ϕ_n *belongs to the interval* $[\underline{\phi}_n, \bar{\phi}_n]$. *Then,*

$$\Delta\mu := \inf_{\mathcal{S}} \liminf_{n\to\infty} \frac{|\gamma(a_{H,n}(\phi_n), a_{L,n}(\phi_n), \phi_n)|}{\phi_n(1-\phi_n)}$$

is such that $\Delta\mu > 0$.

Proof. Suppose by contradiction that $\Delta\mu = 0$ Then, there exists a sequence $\{\phi_n\}_{n\in\mathbf{N}}$ such that $\phi_n \in [\underline{\phi}_n, \bar{\phi}_n]$, and

$$\liminf_{n\to\infty} |\gamma(a_{H,n}(\phi_n), a_{L,n}(\phi_n), \phi_n)| / (\phi_n(1-\phi_n)) = 0.$$

So we can find a subsequence converging to zero. Because by Lemma 3, along this sequence,

$$\lim_{n\to\infty} \left(1 + \phi_n(1-\phi_n)\left(\left|U'_{H,n}(\phi_n)\right| + \left|U'_{L,n}(\phi_n)\right|\right)/r_{S,n}\right) \frac{|\gamma(a_{H,n}(\phi_n), a_{L,n}(\phi_n), \phi_n)|}{\phi_n(1-\phi_n)} > C,$$

if the second term converges to zero, the first must diverge to infinity. But then for n sufficiently large, in equilibrium either both types play the same extremal action, or one player plays an extremal action and the other plays a_θ^*. By Condition 3 and the maintained assumption on the drift function (i.e., $\mu(H, a) \neq \mu(L, a)$, for all $a \in A$), in both cases, for any ϕ, $|\gamma|$ is bounded away from zero, leading to a contradiction. □

Lemma 5. *Suppose that for any ϕ, the speed of learning is equal to $\phi(1-\phi)\Delta\mu$. Then,*

$$\mathbf{E}[\tau] = -\frac{4(1-2\phi_0)\tanh^{-1}(1-2\phi_0)}{(\Delta\mu)^2} + \frac{4(1-2\underline{\phi})(\bar{\phi}-\phi_0)\tanh^{-1}(1-2\underline{\phi})}{(\bar{\phi}-\underline{\phi})(\Delta\mu)^2} + \frac{4(1-2\bar{\phi})(\phi_0-\underline{\phi})\tanh^{-1}(1-2\bar{\phi})}{(\bar{\phi}-\underline{\phi})(\Delta\mu)^2}.$$

Further, as $\{\underline{\phi},\bar{\phi}\}\to\{0,1\}$, for any $r>0$, $\mathbf{E}\left[e^{-r\tau}\right]\to 0$.

Proof. See Online Appendix. □

Proof of Part 1: $r_{DM}>0$, $r_S\to 0$. Consider a sequence $\{r_{S,n}\}_{n=1,2,\dots}$ such that $\lim_{n\to\infty} r_{S,n}=0$ together with a sequence of equilibria and corresponding value functions $\{U_{H,n},U_{L,n}\}$ and cutoffs $\{\underline{\phi}_n,\bar{\phi}_n\}$. We claim that $U_{\theta,n}(\phi)/r_S\to-\infty$ for $\theta\in\{H,L\}$ and $\forall\phi\in[\underline{\phi}_n,\bar{\phi}_n]$. Suppose by contradiction that this is not the case. Denote with $\{\underline{\phi},\bar{\phi}\}$ the limit cutoffs of the DM. Since $r_{DM}>0$, $0<\underline{\phi}<\bar{\phi}<1$. Using an argument similar to Lemma OA.3, one can show that $\{U_{H,n},U_{L,n}\}$ must pointwise converge to $\mathring{U}_1(\phi),\mathring{U}_2(\phi)$, where

$$\mathring{U}_1(\phi) = \frac{(\Pi(H,l)-\Pi(H,h))\underline{\phi}\bar{\phi}}{\phi(\bar{\phi}-\underline{\phi})} + \frac{\bar{\phi}\Pi(H,h)-\underline{\phi}\Pi(H,l)}{\bar{\phi}-\underline{\phi}}$$

$$\mathring{U}_2(\phi) = -\frac{(\Pi(L,l)-\Pi(L,h))(1-\underline{\phi})(1-\bar{\phi})}{(1-\phi)(\bar{\phi}-\underline{\phi})} + \frac{(1-\underline{\phi})\Pi(L,l)-(1-\bar{\phi})\Pi(L,h)}{\bar{\phi}-\underline{\phi}},$$

is the unique solution to the boundary value problem

$$U_1''(\phi) = -2U_1'(\phi)/\phi, \qquad U_2''(\phi) = 2U_2'(\phi)/(1-\phi),$$

under the boundary conditions

$$U(\underline{\phi}) = \begin{pmatrix}\Pi(H,l)\\ \Pi(L,l)\end{pmatrix}, \qquad U(\bar{\phi}) = \begin{pmatrix}\Pi(H,h)\\ \Pi(L,h)\end{pmatrix}.$$

But this implies $\lim_{n\to\infty} U'_{H,n}(\phi)<0$ and $\lim_{n\to\infty} U'_{L,n}(\phi)<0$, contradicting $\lim_{n\to\infty} U'_{\theta,n}(\phi)/r_S\neq-\infty$ for at least one θ. Hence, for any sequence of discount rate $\{r_{I,n}\}_{n=1,2,\dots}$ converging to zero, the sequence of corresponding value functions $\{U_{H,n},U_{L,n}\}$ is such that $\lim_{n\to\infty} U_{H,n}(\phi)/r_S = \lim_{n\to\infty} U_{H,n}(\phi)/r_S = -\infty$, which in turns implies that for r_S low enough the pair of action solving (3) is not interior, for any belief. That is, $a_H(\phi) \to \bar{a}$ and $a_L(\phi) \to \bar{a}$ or $a_H(\phi) \to \underline{a}$ and $a_L(\phi) \to \underline{a}$ for any $\phi \in [\underline{\phi}, \bar{\phi}]$. Whether one or the other case occurs depends on the sign of $\mu(H,a) - \mu(L,a)$, which, by assumption, is independent of a.

Proof of Part 2: $r_S > 0$, $r_{DM} \to 0$. Consider a sequence $\{r_{DM,n}\}_{n\in\mathbf{N}}$ such that $\lim_{n\to\infty} r_{DM,n} = 0$. together with a sequence of equilibria and corresponding value functions $\{U_{H,n}, U_{L,n}\}$ and cutoffs $\{\bar{\phi}_n, \underline{\phi}_n\}$.

First, we show that $\{\bar{\phi}_n, \underline{\phi}_n\} \to \{0,1\}$. By Lemma 4, there exists a strictly positive lower bound to the difference in conditional drifts in the sequence of games. Consider the sequence of optimal stopping problems for the DM associated to the discount rate $\{r_{DM,n}\}_{n\in\mathbf{N}}$ in which the speed of learning is, for any ϕ, $\phi(1-\phi)\Delta\mu$, from Lemma 4. It follows an adaptation of Moscarini and Smith (2003)'s argument (see their proof of Proposition 5(e)), that the cutoff shift out strictly as the discount rate increases. As a result, in the limit, they converge to 0 and 1, respectively. A fortiori, $\{\bar{\phi}_n, \underline{\phi}_n\} \to \{0,1\}$.

Second, we show that the $\lim_{n\to\infty} U_{\theta,n}(\phi) = \pi^*(\theta)$, for all $\phi \in (0,1)$. For any n, the payoff of type θ of the sender can be written as,[22]

$$
\begin{aligned}
U_{\theta,n}(\phi) = \mathbf{E}_\theta[e^{-r_S\tau} \mid \phi_\tau = \underline{\phi}, \phi_0 = \phi]\Pr{}_\theta[\phi_\tau = \underline{\phi} \mid \phi_0 = \phi]\Pi(\theta, l) \\
+ \mathbf{E}_\theta[e^{-r_S\tau} \mid \phi_\tau = \bar{\phi}, \phi_0 = \phi]\Pr{}_\theta[\phi_\tau = \bar{\phi} \mid \phi_0 = \phi]\Pi(\theta, h) \\
+ \mathbf{E}_\theta\left[\int_0^\tau r_S e^{-r_S t}\pi(\theta, a(\phi_t))\,\mathrm{d}t\right]. \quad (8)
\end{aligned}
$$

By Lemma 5 and Lemma 4, the first two terms vanish. As a result, the limit value functions reduce to the limit of the last term in (8). By Theorem 3.7 of Stokey (2009), we have

$$
\begin{aligned}
\mathbf{E}_\theta\left[\int_0^\tau r_S e^{-r_S t}\pi(\theta, a(\phi_t))\,\mathrm{d}t \mid \phi_0 = \hat{\phi}\right] &= \mathbf{E}\left[\int_{\underline{\phi}}^{\bar{\phi}} r_S \pi(\theta, a(\phi))\hat{\ell}(\phi; \hat{\phi}, \tau; r_S)\,\mathrm{d}\phi\right] \\
&= \int_{\underline{\phi}}^{\bar{\phi}} r_S \pi(\theta, a(\phi))\hat{L}(\phi; \hat{\phi}, \underline{\phi}, \bar{\phi}; r_S)\,\mathrm{d}\phi
\end{aligned}
$$

[22] The expectation in the last term in the equation below also depends on $\underline{\phi}_n$ and $\bar{\phi}_n$ but we are omitting this dependence for notational convenience.

where $\hat{\ell}$ denotes the discounted local time function evaluated at ϕ and $\hat{L}(\phi;\hat{\phi},\underline{\phi},\overline{\phi};r_S) \coloneqq \mathbf{E}[\hat{\ell}(\phi;\hat{\phi},\tau;r_S)]$.

As in the proof of Lemma OA.3, by Arzelà-Ascoli theorem (Ok, 2007, Chapter D.6), the sequence of pairs of value functions has a converging subsequence and the limit pair is a continuously differentiable function that solves the limit boundary value problem. To put it differently, the limit pair of value functions solve the limit best-reply problem. On the other hand, in the limit, each type of sender maximizes

$$\max_{a_\theta(\phi)\in A^{[0,1]}} \int_0^1 r_S \pi(\theta, a_\theta(\phi))\hat{L}(\phi;\hat{\phi},\underline{\phi},\overline{\phi};r_S)\,\mathrm{d}\phi,$$

where the choice of action affects not only the payoff but also the expected discounted local time function. Clearly, the expected discounted local time function is affected not only by type θ's action but also by the other type, as well as by the DM's conjecture. However, we can look at a relaxed problem in which the only constraint on the choice of the expected discounted local time function is

$$\int_0^1 \hat{L}(\phi;\hat{\phi},\underline{\phi},\overline{\phi};r_S)\,\mathrm{d}\phi = 1/r_S.$$

Recall that one can interpret the expected discounted local time function as a weighting function, similar to a density, with the difference, that integrates to $1/r_S$.

Since π is single-peaked, it is then immediately that is optimal to choose the myopically optimal action at any time, that is, $a_\theta(\phi) = a^*_\theta$ for any $\phi \in (0,1)$.

Proof of Part 3: $\lim_{n\to\infty} r_{S,n}/r_{DM,n} \to k \in (0,\infty)$. For each $n = 1, 2, \ldots$, we consider an equilibrium associated with the discount factors $r_{DM,n}$ and $r_{S,n}$together with a sequence of equilibria and corresponding value functions $\{U_{H,n}, U_{L,n}\}$ and cutoffs $\{\underline{\phi}_n, \overline{\phi}_n\}$. Assume that $\lim_{n\to\infty} r_{S,n} = 0$ and $\lim_{n\to\infty} r_{DM,n} = 0$ and $\lim_{n\to\infty}(r_{S,n}/r_{DM,n}) = \kappa \in (0,\infty)$. Recall that by Lemma 4, there exists a strictly positive lower bound to the difference in conditional drifts in the sequence of games.

Lemma 6. *For each n, consider the sequence of DM's optimal stopping problems when the difference in conditional drifts is, for each ϕ, $\Delta\mu\phi(1-\phi)$, from Lemma 4. Denote the value function in these decision problems with $V_n^\dagger$ and the decision time* *with $\tau_n^\dagger$. Then,*

$$\lim_{n\to\infty} V_n^\dagger(\phi_0) = \phi_0 G(H,h) + (1-\phi_0)\, G(L,l),$$
$$\lim_{n\to\infty} \mathbf{E}\left[e^{-r_{DM,n}\tau_n^\dagger}\right] = 1, \qquad \textit{and} \qquad \lim_{n\to\infty} \mathbf{E}\left[\tau_n^\dagger\right] = \infty.$$

Proof. From the proof of Proposition 2, the value function admits the following characterization,

$$V_n^\dagger(\phi) = \tilde{A}(1-\phi)^{\left(1+\sqrt{1+8r_{DM_n}\sigma^2/\overline{\Delta\mu}^2}\right)/2}(\phi)^{\left(1-\sqrt{1+8r_{DM_n}\sigma^2/\overline{\Delta\mu}^2}\right)/2} + \tilde{B}(1-\phi)^{\left(1-\sqrt{1+8r_{DM_n}\sigma^2/\overline{\Delta\mu}^2}\right)/2}(\phi)^{\left(1+\sqrt{1+8r_{DM_n}\sigma^2/\overline{\Delta\mu}^2}\right)/2}, \tag{9}$$

for any $\phi \in (0,1)$ such that $V_n^\dagger(\phi) > \max\{\phi G(H,h) + (1-\phi)G(L,h), \phi G(H,l) + (1-\phi)G(L,l)\}$. In the limit, as $r_{DM_n} \to 0$, both fundamental solutions become linear in ϕ. It follows that $V_n^\dagger(\phi)$, which by standard results is convex and bounded below by $\max\{\phi G(H,h) + (1-\phi)G(L,h), \phi G(H,l) + (1-\phi)G(L,l)\}$ and above by $\phi G(H,h) + (1-\phi)\, G(L,l)$, converges to $\phi G(H,h) + (1-\phi)\, G(L,l)$. But then, smooth pasting can hold only if the cutoffs characterizing the optimal policy converge to 0 and 1. For the value function to converge to the complete information value, the cost of delay must converge to zero, that is, $\lim_{n\to\infty} \mathbf{E}\left[e^{-r_{DM,n}\tau_n^\dagger}\right] = 1$. To show that $\mathbf{E}\left[\tau_n^\dagger\right] = \infty$, one follow the same steps as Lemma 5. □

Lemma 6 implies, a fortiori, that

$$\lim_{n\to\infty} V_n(\phi_0) = \phi_0 G(H,h) + (1-\phi_0)\, G(L,l),$$
$$\lim_{n\to\infty} \mathbf{E}\left[e^{-r_{DM,n}\tau_n}\right] = 1, \qquad \text{and} \qquad \lim_{n\to\infty} \mathbf{E}\left[\tau_n\right] = \infty,$$

In turns, by Claim OA.6 in Ekmekci et al. (2022), $\lim_{n\to\infty} \mathbf{E}\left[e^{-r_{S,n}\tau_n}\right] = 1$. Given that π is bounded, it follows that, $\mathbf{E}\left[\int_0^\tau e^{-r_{S,n}t}\pi(\theta, a(\phi_t))\,\mathrm{d}t\right] = 0$. As a result, the value function of the sender converges to

$$\lim_{n\to\infty} U_{H,n}(\phi_n) = \Pi(H,h) \qquad \lim_{n\to\infty} U_{L,n}(\phi_n) = \Pi(L,l) \qquad \phi \in (0,1). \tag{10}$$

Next, we show that $\lim_{n\to\infty} U'_{\theta,n}(\phi_n)/r_{S,n} = -\infty$. For each n, consider the unique solutions to to the boundary value problem

$$U_1''(\phi) = -2U_1'(\phi)/\phi, \qquad U_2''(\phi) = 2U_2'(\phi)/(1-\phi),$$

under the boundary conditions

$$\begin{aligned} U_1'(\underline{\phi}_n) &= U'_{H,n}(\underline{\phi}_n), & U_2'(\bar{\phi}_n) &= U'_{L,n}(\bar{\phi}_n). \\ U_1(\underline{\phi}_n) &= \Pi(H,l), & U_2(\bar{\phi}_n) &= \Pi(L,h). \end{aligned}$$

For any $n > 0$, the boundary values are bounded away from zero, by (10), and the unique solution to the above boundary problem is

$$\begin{aligned} \mathring{U}_{1,n}(\phi) &= -\frac{(U'_{H,n}(\underline{\phi}_n)\underline{\phi}_n)^2}{\phi} + \Pi(H,l) + U'_{H,n}(\underline{\phi}_n)\underline{\phi}_n, \\ \mathring{U}_{2,n}(\phi) &= \frac{(U'_{L,n}(\bar{\phi}_n)(1-\bar{\phi}_n))^2}{1-\phi} + \Pi(L,h) + U'_{L,n}(\bar{\phi}_n)(1-\bar{\phi}_n). \end{aligned}$$

By Lemma OA.3, the sequence of value functions $\{U_{H,n}, U_{L,n}\}$ and their first derivatives converge to $\{\mathring{U}_{H,n}, \mathring{U}_{L,n}\}$ and their first derivatives. By Lemma OA.4,

$$\lim_{n\to\infty}(1-\bar{\phi}_n)^2/r_{S,n} = \lim_{n\to\infty}(\underline{\phi}_n)^2/r_{S,n} = \infty,$$

which implies that $\lim_{n\to\infty} U'_{\theta,n}(\phi_n)/r_{S,n} = -\infty$.

8.4 Proof for Section 5 and Section 6

See Online Appendix.